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Master's Thesis in Arithmetic Geometry

Tropical Cycle Classes for the Berkovich Analytification of Algebraic Varieties

Jonathan Pirnay

Advisor: Prof. Dr. Klaus Künnemann
Secondary Advisor: M.Sc. Martino Stoffel

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1

Introduction

In complex geometry there is a natural decomposition of the complex differential k-forms A^k on a complex manifold M into (p,q)-forms, i.e.

$$A^k = \bigoplus_{p+q=k} A^{p,q}.$$

These (p,q)-forms yield a finer geometric structure on M than the k-forms and are hence fundamental objects of study in complex differential geometry. The aim of this thesis is to introduce and study real-valued (p,q)-differential forms on the Berkovich analytification X^{an} of an n-dimensional algebraic variety X (that is, an integral separated K-scheme of finite type) over an algebraically closed, nontrivially valued, complete non-Archimedean field K. Here the analytification X^{an} takes the role of the complex manifold M. This was done first by Chambert-Loir and Ducros in [CLD12] in the more general setting of a good K-analytic space Y (analytic in the sense of Berkovich [Ber90]). Their general strategy is to tropicalize analytic moment maps and then pull back superforms as introduced by Lagerberg [Lag12]. Here an analytic moment map is an analytic morphism $f\colon Y \to T^{\mathrm{an}}$, where $T^{\mathrm{an}} = (\mathbb{G}_m^q)^{\mathrm{an}}$ is the Berkovich analytification of a (split) multiplicative torus $\mathbb{G}_m^q = \mathrm{Spec}(K[T_1^{\pm 1}, \ldots, T_q^{\pm 1}])$. They obtain sheaves $A_Y^{p,q}$ of differential forms on Y with differential operators d' and d'', which can be regarded as an analogue to the differential operators ∂ and $\overline{\partial}$ on a complex manifold.

However we will follow the more algebraic strategy by Gubler as in [Gub16]. His approach is to cover the algebraic variety X with *very affine* open subsets, i.e. affine opens that allow a closed immersion into some (split) muliplicative torus. Broadly speaking, the sheaf of (p,q)-differential forms $A^{p,q}_{X^{\mathrm{an}}}$ on X^{an} is then obtained by locally pulling back superforms along the tropicalization of the analytification of such closed immersions.

Furthermore, in the complex geometric situation, Dolbeault's theorem states that the complex $(A^{p,\bullet}(M), \overline{\partial})$ computes the sheaf cohomology groups of the sheaf of holomorphic differential forms, i.e. that

$$\mathrm{H}^{p,q}(M) := \mathrm{H}^q(A^{p,\bullet}(M)) \cong \mathrm{H}^q(M, \Omega^p),$$

where Ω^p is the sheaf of holomorphic p-forms on the complex manifold M. In our setting, an analogue of the Poincaré-Lemma proved by Jell in [Jel16a] yields that the Dolbeault cohomology

$$\mathrm{H}^{p,q}(X^{\mathrm{an}}) := \frac{\ker(d''\colon A^{p,q}_{X^{\mathrm{an}}}(X^{\mathrm{an}}) \to A^{p,q+1}_{X^{\mathrm{an}}}(X^{\mathrm{an}}))}{\mathrm{im}(d''\colon A^{p,q-1}_{X^{\mathrm{an}}}(X^{\mathrm{an}}) \to A^{p,q}_{X^{\mathrm{an}}}(X^{\mathrm{an}}))} \tag{1.0.1}$$

on the analytification X^{an} computes the sheaf cohomology of $\ker(d''\colon A^{p,0}_{X^{\mathrm{an}}} \to A^{p,1}_{X^{\mathrm{an}}})$. One consequence of this is that $\mathrm{H}^\bullet(X^{\mathrm{an}}, \underline{\mathbb{R}}) \cong \mathrm{H}^{0,\bullet}(X^{\mathrm{an}})$ for the constant sheaf associated to $\mathbb{R}$.

Liu has used the Dolbeault cohomology groups in his preprint [Liu17] to define a *tropical cycle class map*

$$\mathrm{CH}^p(X) \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathrm{H}^{p,p}(X^{\mathrm{an}})$$

which connects the Chow group $\mathrm{CH}^p(X)$ of a smooth variety X with the differential forms on its analytification. The general aim in this thesis is to develop the necessary tools and ideas to understand the construction of the tropical cycle class map and finally formulate a tropical version of the Cauchy formula in multi-variable complex analysis [Liu17, Theorem 3.7].

We now outline the structure of the thesis in detail. We start purely algebro-geometric: The tropical cycle class map utilizes Bloch's formula, i.e. the result that on smooth varietes we have for any $p \geq 0$ an isomorphism $\mathrm{CH}^p(X) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathrm{H}^p(X, \mathscr{K}^p_X)$, where $\mathscr{K}^p_X$ is the p-th sheaf of rational Milnor K-Theory of the structure sheaf $\mathcal{O}_X$. Hence the aim in Chapter 2 is to introduce the tools to formally understand where this isomorphism comes from (however we do not prove the exactness of the complex which leads to Bloch's formula). We give a short recollection of algebraic cycles, rational equivalence and the construction of the Chow groups for an algebraic variety X in Section 2.1.

In Section 2.2 we define the Milnor K-groups $K^M_n(R)$ of a ring R which are defined as the n-th tensor product $(R^*)^{\otimes n}$ on the multiplicative unit group modulo the Steinberg relation. One object of interest will be the *tame symbol*, a unique homomorphism

$$K^M_n(K) \to K^M_{n-1}(\kappa)$$

for a discrete valuation field K with residue field κ. With the tame symbol we can define residue morphisms d and Kato showed that for every $p \geq 0$ we obtain a complex of abelian groups

$$\bigoplus_{x\in X^{(0)}} K_p^M(k(x)) \xrightarrow{d} \bigoplus_{x\in X^{(1)}} K_{p-1}^M(k(x)) \xrightarrow{d} \cdots \xrightarrow{d} \bigoplus_{x\in X^{(p)}} K_0^M(k(x)) \to 0,$$

called the *Gersten complex*, for a smooth n-dimensional variety X over a field K. We will see that the p-th Chow group $\mathrm{CH}^p(X)$ arises naturally as the last cokernel in the Gersten complex. The presentation of the material in Section 2.2 mainly follows Chapter 7 and 8 of the book [GS17] by Gille and Szamuely. In Section 2.3 we examine for a ringed space $(X, \mathcal{O}_X)$ the q-th sheaf $\mathscr{K}_X^q$ of rational Milnor K-Theory, which is defined as the sheafification of the presheaf $U \mapsto K_q^M(\mathcal{O}_X(U)) \otimes_Z \mathbb{Q}$. In particular we will be able to formulate Bloch's formula for a smooth variety X, and furthermore examine how the sheaf cohomology with support in Z behaves for a closed subvariety Z of X. For understanding the tropical cycle class map in the final Theorem 4.2.5, it is useful to give another description of the image of the class $[Z] \in \mathrm{CH}^p(X)$ of a smooth closed subvariety Z of X of codimension p under the isomorphism $\mathrm{CH}^p(X) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathrm{H}^p(X, \mathscr{K}_X^p)$ in terms of regular sequences and Čech cohomology. The very technical Section 2.4 is fully dedicated to this description. This section and Section 2.3 are detailed accounts of the explanations in Section 2 of [Liu17].

Chapter 3 leaves the Chow groups for a moment and introduces differential forms on the analytification. This chapter forms the heart of the thesis, and outlines the material in [Gub16] in much detail. We start in Section 3.1 with the construction of superforms on $\mathbb{R}^r$ as introduced by Lagerberg in [Lag12], which are analogues of real-valued (p, q)-forms on complex manifolds. The space of superforms of bidegree (p, q) on an open subset $U \subseteq \mathbb{R}^r$ is formally defined as the tensor product

$$A^{p,q}(U) := A^p(U) \otimes_{C^\infty(U)} A^q(U),$$

where $A^p(U)$ is the space of smooth real differential forms of degree p. As in differential geometry, one can define differential operators $d' \colon A^{p,q}(U) \to A^{p+1,q}(U)$ and $d'' \colon A^{p,q}(U) \to A^{p,q+1}(U)$, as well as pullbacks with respect to affine maps. We introduce the notion of integration of (r, r)-superforms with compact support and get an analogue of the classical change of variables formula 3.1.10, in particular we see that integration only depends on the underlying integral $\mathbb{R}$-affine structure of $\mathbb{R}^r$, but not on a chosen orientation. Furthermore we obtain an analogue of Stokes' theorem on integral $\mathbb{R}$-affine polyhedra. In Section 3.2 we extend the notion of superforms and integration to weighted polyhedral complexes.

In Section 3.3 we explain Gubler's approach of covering X^{an} with tropical charts.
For this let K be an algebraically closed field endowed with a complete nontrivial non-Archimedean absolute value $|\cdot|_K$ (in particular the associated residue field $\tilde{K}$ is algebraically

closed as well). Furthermore let X be a variety over K with Berkovich analytification X^{an}. Let $\nu := -\log|\cdot|_K$ be the valuation associated to $|\cdot|_K$ and $\Gamma := \nu(K^*) \subseteq \mathbb{R}$ its value group. In Remark 3.3.1 we briefly recall the basic facts needed in this thesis about the Berkovich analytification. One thing to note is that if $X = \mathrm{Spec}(A)$ is affine, then one can consider X^{an} as the set of multiplicative seminorms on A extending $|\cdot|_K$. We can analytify the split multiplicative torus $T = \mathbb{G}_m^r = \mathrm{Spec}(K[T_1^{\pm 1}, \ldots, T_r^{\pm 1}])$ and obtain a *tropicalization map*

$$\mathrm{trop}\colon T^{\mathrm{an}} \to \mathbb{R}^r, \quad p \mapsto (-\log|T_1(p)|, \ldots, -\log|T_r(p)|).$$

Here $|T_i(p)| = p(T_i)$ for a multiplicative seminorm $p \in T^{\mathrm{an}}$. By the Bieri-Groves-Theorem, for a closed subvariety Y of T, the set $\mathrm{Trop}(Y) = \mathrm{trop}(Y^{\mathrm{an}})$ is an integral Γ-affine polyhedral complex, which – after appropriate choice of weights – is even balanced, i.e. a tropical cycle. We will see that the very affine open subsets of X form a basis for the Zariski topology. An open $U \subseteq X$ is very affine if it allows a closed immersion $\varphi_U\colon U \to \mathbb{G}_m^r$ into some split multiplicative torus. The idea is then to cover X^{an} with *tropical charts* (V, φ_U), where $V \subseteq U^{\mathrm{an}}$ is an open subset of some very affine open U, and V arises as a preimage of an open subset of the support of the polyhedral complex $\mathrm{Trop}(U)$ under the tropicalization map.

In Section 3.4 we define differential forms on the analytification X^{an} in a local way by pulling back superforms on the polyhedral complexes $\mathrm{Trop}(U)$ along tropical charts (V, φ_U), where the superforms have to agree on intersections in a specific sense. This definition is by nature local and so for any $p, q \geq 0$ we obtain a sheaf of differential forms $A^{p,q}_{X^{\mathrm{an}}}$ on the paracompact Hausdorff space X^{an}, which is a fine sheaf. We often omit the subscript if the ambient variety X is understood and only write $A^{p,q}$. In the rest of the section we study differential forms in more detail. For example interestingly the support of an $\alpha \in A^{p,q}(X^{\mathrm{an}})$ is 'quite small' and is already contained in the fiber of the generic point of X under the analytification morphism $\pi\colon X^{\mathrm{an}} \to X$ (see Remark 3.4.15). In particular we will see in Proposition 3.4.16 that if α has compact support, it is already defined by a single superform on $\mathrm{Trop}(U)$ for a very affine open $U \subseteq X$, which leads to a well-defined notion of integration of α.
We conclude the section by looking at the complex of sheaves

$$0 \longrightarrow A^{p,0} \xrightarrow{d''} A^{p,1} \xrightarrow{d''} \ldots \xrightarrow{d''} A^{p,n} \longrightarrow 0$$

on X^{an}. By the d''-Poincaré Lemma by Jell in [Jel16a], this complex is exact in positive degrees, and as the sheaves are fine (hence acyclic) we obtain isomorphisms

$$\mathrm{H}^{\bullet}(X^{\mathrm{an}}, \ker(d''\colon A^{p,0} \to A^{p,1})) \cong \mathrm{H}^{p,\bullet}(X^{\mathrm{an}}),$$

where the right hand side denotes the Dolbeault cohomology groups as in (1.0.1) above.

In Chapter 4 we are going to describe the tropical cycle class map as introduced in Section 3 in [Liu17]. For this we first define in Section 4.1 a $\mathbb{Q}$-linear map of sheaves

$$\tau^p_{X^{\mathrm{an}}} \colon \mathscr{K}^p_{X^{\mathrm{an}}} \to \ker(d''\colon A^{p,0}_{X^{\mathrm{an}}} \to A^{p,1}_{X^{\mathrm{an}}})$$

which canonically induces a rational subsheaf

$$\mathscr{T}^p_{X^{\mathrm{an}}} = \mathscr{K}^p_{X^{\mathrm{an}}} / \ker \tau^p_{X^{\mathrm{an}}}$$

of $\ker(d''\colon A^{p,0}_{X^{\mathrm{an}}} \to A^{p,1}_{X^{\mathrm{an}}})$. By one of our main results (see Proposition 4.1.8) in this section there is an isomorphism $\mathscr{T}^p_{X^{\mathrm{an}}} \otimes_{\mathbb{Q}} \mathbb{R} \cong \ker(d''\colon A^{p,0}_{X^{\mathrm{an}}} \to A^{p,1}_{X^{\mathrm{an}}})$ and hence

$$\mathrm{H}^q(X^{\mathrm{an}}, \mathscr{T}^p_{X^{\mathrm{an}}}) \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathrm{H}^{p,q}(X^{\mathrm{an}})$$

for all $p, q \geq 0$, which endows the Dolbeault cohomology groups with a canonical rational structure.
Finally in Section 4.2 we introduce the tropical cycle class map as the composition

$$\mathrm{CH}^p(X)_{\mathbb{Q}} \cong \mathrm{H}^p(X, \mathscr{K}^p_X) \longrightarrow \mathrm{H}^p(X^{\mathrm{an}}, \mathscr{K}^p_{X^{\mathrm{an}}}) \longrightarrow \mathrm{H}^p(X^{\mathrm{an}}, \mathscr{T}^p_{X^{\mathrm{an}}}),$$

where we regard $\mathrm{H}^p(X^{\mathrm{an}}, \mathscr{T}^p_{X^{\mathrm{an}}})$ as a rational subspace of $\mathrm{H}^{p,p}(X^{\mathrm{an}})$.
For a d''-closed differential form $\omega \in A^{n-p,n-p}_{X^{\mathrm{an}},c}(X^{\mathrm{an}})$ with compact support and a cohomology class $[\eta] \in \mathrm{H}^{p,p}(X)$ given by a representative $\eta \in \ker(d''\colon A^{p,p}(X^{\mathrm{an}}) \to A^{p,p+1}(X^{\mathrm{an}}))$ there is a well-defined integral

$$\int_{X^{\mathrm{an}}} [\eta] \wedge \omega := \int_{X^{\mathrm{an}}} \eta \wedge \omega,$$

and we obtain the following theorem by Liu.

Theorem (4.2.5) *Let X be a smooth variety over K of dimension n. For every algebraic cycle Z of X of codimension p, and any d''-closed differential form $\omega \in A^{n-p,n-p}_{X^{\mathrm{an}},c}(X^{\mathrm{an}})$ with compact support, we have the equality*

$$\int_{X^{\mathrm{an}}} \mathrm{cl}_{\mathrm{trop}}(Z) \wedge \omega = \int_{Z^{\mathrm{an}}} \omega.$$

Here, if we formally write $Z = \sum_{i=1}^k a_i Z_i$ with $a_i \in \mathbb{Z}$ and the Z_i's are closed subvarieties of codimension p, we define

$$\int_{Z^{\mathrm{an}}} \omega := \sum_{i=1}^k a_i \int_{Z_i^{\mathrm{an}}} \omega.$$

Although we do not fully prove the theorem, we conclude the thesis by an explanation of how the special description of Bloch's formula via Čech cohomology and regular sequences is used in Liu's proof.

Appendix A contains detailed background for sheaf cohomology with support, by following Exercise II.1.20 and Exercise III.2.3 in [Har77].

Appendix B is a review of Čech cohomology with particular emphasis on an explicit description of the canonical map from Čech cohomology to sheaf cohomology of a sheaf $\mathcal{F}$, provided we have a resolution $0 \to \mathcal{F} \to \mathcal{I}^\bullet$ of $\mathcal{F}$ where the sheaves $\mathcal{I}^\bullet$ are acyclic in both Čech and sheaf cohomology.

Terminology. All rings are supposed to be commutative and with 1. For a ring R, we denote its multiplicative group of units by R^*. For an element $f \in R$ we write $R[\frac{1}{f}]$ for the localization away from f. The nonvanishing locus of f is denoted by $D(f) \subseteq \operatorname{Spec}(R)$, and the vanishing locus of an ideal $\mathfrak{a} \subseteq R$ by $V(\mathfrak{a})$. For a locally ringed space X we usually denote its structure sheaf by $\mathcal{O}_X$ and its stalks at a point $x \in X$ by $(\mathcal{O}_{X,x}, \mathfrak{m}_{X,x})$ with residue field $k(x)$. For a non-Archimedean field K with absolute value $|\cdot|$, put

$$K^\circ = \{x \in K \mid |x| \leq 1\}, \quad K^{\circ\circ} = \{x \in K \mid |x| < 1\}$$

and write $\tilde{K}$ for the residue field $K^\circ/K^{\circ\circ}$. For a topological space X and the global sections functor $\Gamma(X, \cdot)$ from abelian sheaves on X to abelian groups, we denote the right derived functors of $\Gamma(X, \cdot)$ by $\mathrm{H}^i(X, \cdot)$, which assign the i-th sheaf cohomology group $\mathrm{H}^i(X, \mathcal{F})$ to a sheaf $\mathcal{F}$. Finally, for a K-vector space V, we write V^* for its associated dual space.

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2

The Chow Group and Bloch's Formula

This chapter develops the necessary language to formulate Bloch's formula, an isomorphism which relates the Chow group of a smooth variety X over a field to the cohomology of X with coefficients in the Milnor K-Theory of the structure sheaf $\mathcal{O}_X$.

2.1 Chow groups

In this section we briefly review the construction of Chow groups of a variety X, and mainly follow the material in [Ful98, 1.2 and 1.3]. In the following let X be a variety of finite dimension n, i.e. an integral (hence reduced and irreducible), separated scheme of finite type over an algebraically closed field K. Denote the generic point of X by η, i.e. $X = \overline{\{\eta\}}$, and the function field of X by $K(X) = \mathcal{O}_{X,\eta}$. The theory works in more generality, however we restrict ourselves to X being a variety, in particular X is clearly equidimensional (as it is irreducible).

Definition 2.1.1.

i.) Denote by $X_{(p)}$ (resp. $X^{(p)}$) the set of points of X of dimension (resp. codimension) p. Recall that the codimension of a point $x \in X$ is given by the Krull dimension of its local ring $\mathcal{O}_{X,x}$.

ii.) An *algebraic cycle* of X is a finite formal sum $\sum n_i V_i$, where the V_i are closed subvarieties of X, and the n_i are integers. The cycles form a group, which we denote by

$Z(X)$, or just Z if the ambient variety X is understood. Since we have a bijection

$$\{\text{irreducible reduced closed subschemes } V \subseteq X\} \longleftrightarrow \{\text{points of } X\}$$

(V maps to its generic point, and in converse a point maps to its closure with its unique reduced induced subscheme structure), we can also write

$$Z = \bigoplus_{x \in X} \mathbb{Z}.$$

In general for a point $x \in X$ we will always consider the closure $\overline{\{x\}}$ with its structure as a closed subvariety.

iii.) We grade Z by dimension, by letting Z_p denote the algebraic cycles of dimension p, i.e. $Z_p := \bigoplus_{x \in X_{(p)}} \mathbb{Z}$. Additionally, we may grade Z by codimension as well, writing $Z^p := \bigoplus_{x \in X^{(p)}} \mathbb{Z}$. In the following we will mostly use Z^p, however note that as X is an n-dimensional variety, we have

$$Z_{n-p} = Z^p.$$

iv.) For a closed subvariety $V \subseteq X$, we call its corresponding cycle $V \in Z$ a *prime cycle.*

Definition 2.1.2. Let $V = \overline{\{y\}}$ be a closed subvariety of X of codimension one. The local ring $(A, \mathfrak{m}) = \mathcal{O}_{X,y}$ is a local domain of dimension one with $\mathrm{Frac}(A) = K(X)$. Let $f \in K(X)^*$. We will now define the notion of *order of vanishing* $\mathrm{ord}_V(f) \in \mathbb{Z}$ of f along V, which will be a group homomorphism $K(X)^* \to \mathbb{Z}$.

As $K(X) = \mathrm{Frac}(A)$, we can write $f = \frac{a}{b}$ for $a, b \in A$, hence it suffices to define $\mathrm{ord}_V(f)$ for $f \in A$. First suppose that X is non-singular along V, i.e. A is a discrete valuation ring (this is for example the case if the variety X is normal). Then $f = u \cdot t^m$, for $u \in A^*$, a generator t of the maximal ideal $\mathfrak{m}$, and an integer m. In this case we set $\mathrm{ord}_V(f) := m$ and it is clear that this is indeed a group homomorphism.

In the general case, when X is singular along V, we define

$$\mathrm{ord}_V(f) = l_A(A/(f)), \tag{2.1.1}$$

where the right hand side denotes the length of the A-module $A/(f)$.

Remark 2.1.3. Let $\varphi\colon \tilde{X} \to X$ be the normalization of X in its function field. Note that φ is a finite morphism, as X is a variety. Hence for a closed subvariety $V = \overline{\{y\}}$ of codimension one, the fiber $\varphi^{-1}(\{y\}) = \{\tilde{y}_1, \dots, \tilde{y}_m\}$ is finite with corresponding closed subvarieties $\tilde{V}_i = \overline{\{\tilde{y}_i\}}$. Note that $\varphi(\tilde{V}_i) = V$ as φ is finite and hence closed. Furthermore for each of these $\tilde{V}_i$ we have a finite field extension $K(\tilde{V}_i)/K(V)$.

Now let $f \in K(X)^* = K(\tilde{X})^*$. By [Ful98, A.3.1] we obtain the equality

$$\mathrm{ord}_V(f) = \sum \mathrm{ord}_{\tilde{V}}(f)[K(\tilde{V}) : K(V)], \tag{2.1.2}$$

where the sum on the right hand side is taken over all closed subvarieties $\tilde{V} = \overline{\{\tilde{y}\}}$ of $\tilde{X}$ which map onto V (hence the order $\mathrm{ord}_{\tilde{V}}(f)$ can be taken via the discrete valuation ring $\mathcal{O}_{\tilde{X},\tilde{y}}$) and $[K(\tilde{V}) : K(V)] < \infty$ is the degree of the field extension. This shows that (2.1.1) is truly a generalization of the non-singular case.

Definition 2.1.4. (Chow group)

i.) Let W be $(k+1)$-dimensional closed subvariety of X, and $f \in K(W)^*$. We define a k-cycle $\mathrm{div}(f)$ on X by setting

$$\mathrm{div}(f) = \sum \mathrm{ord}_V(f)V,$$

where the sum is taken over all k-dimensional (i.e. with codimension one in W) closed subvarieties V of W. This sum is indeed finite: As known, for a Noetherian integral scheme W and $f \in K(W)^*$, we have $f \in \mathcal{O}^*_{W,y}$ for almost all points $y \in W$ of codimension one.

ii.) As above let Z^k be the free abelian group generated by codimension-k subvarieties of X. Now let $B^k(X) \subseteq Z^k(X)$ be the subgroup generated by all the $\mathrm{div}(f)$ where $f \in K(W)^*$ for an arbitrary $(k-1)$-codimensional closed subvariety W. We set

$$\mathrm{CH}^k(X) := Z^k(X)/B^k(X)$$

and call $\mathrm{CH}^k(X)$ the *Chow group* of codimension-k cycles on X. An element in $B^k(X)$ is said to be *rationally equivalent to zero.*

iii.) For a $V \in Z^k(X)$ we write $[V]$ for its image in $\mathrm{CH}^k(X)$.

Example 2.1.5. i.) We have $\mathrm{CH}^p(X) = 0$ for $p > n = \dim(X)$.

ii.) The only point of codimension 0 is the generic point η of X, hence $\mathrm{CH}^0(X)$ is freely generated by $X \in Z^0(X)$.

iii.) $\mathrm{CH}^1(X)$ is equal to the class group of Weil divisors on X. E.g. if X is smooth, this can be identified with the Picard group $\mathrm{Pic}(X)$.

The examples given above are all too simple: for $2 \leq p \leq \dim X$ the Chow groups are in general very hard to compute.

2.2 Milnor K-Theory

In this section we define Milnor K-groups and state the two basic constructions in the theory: the *tame symbol* and *norm maps*, where the latter generalize the field norm for finite field extensions to higher K-groups. These main concepts will later be used to formulate the ideas for Bloch's formula. We follow the explanations in Chapter 7 and 8 of [GS17].

Definition 2.2.1. Let R be a ring. For $n \geq 0$ we define the *n-th Milnor K-group* $K_n^M(R)$ as the quotient of the n-th tensor power $(R^*)^{\otimes n}$ of the multiplicative group of units of R by the subgroup generated by those elements $a_1 \otimes \cdots \otimes a_n$ for which $a_i + a_j = 1$ in R for some $1 \leq i < j \leq n$. Hence $K_0^M(R) = \mathbb{Z}$ and $K_1^M(R) = R^*$. The relation $a_i + a_j = 1$ is called the *Steinberg relation*.

We write $\{a_1, \ldots, a_n\}$ for the image $a_1 \otimes \cdots \otimes a_n$ in $K_n^M(R)$, and adopt the convention to set $K_n^M(R) = 0$ for $n < 0$.

Some words of caution on the use of notation: We use multiplicative notation for the group operation on R^*, however use additive notation for $K_n^M(R)$. This means for example that we have $\{a, b\} + \{a, c\} = \{a, b \cdot c\}$ for $a, b, c \in R^*$.

Although we will later work over arbitrary rings, we restrict ourselves now to the case $R = k$, where k is an arbitrary field.

Remark 2.2.2. There is a natural product structure

$$K_n^M(k) \times K_m^M(k) \to K_{n+m}^M(k),\ (a, b) \mapsto \{a, b\} \tag{2.2.1}$$

coming from the tensor product pairing $(k^*)^{\otimes n} \times (k^*)^{\otimes m} \to (k^*)^{\otimes n+m}$ which clearly preserves the Steinberg relation. Furthermore this product operation equips the direct sum

$$K_*^M(k) = \bigoplus_{n \geq 0} K_n^M(k)$$

with the structure of a graded ring indexed by the nonnegative integers.

Proposition 2.2.3. Let k be a field.

i.) The group $K_2^M(k)$ satisfies the relations

$$\{x, -x\} = 0 \text{ and } \{x, x\} = \{x, -1\}.$$

ii.) The product operation (2.2.1) is graded-commutative, i.e. it satisfies

$$\{a, b\} = (-1)^{mn}\{b, a\}$$

for all $a \in K_n^M(k)$ and $b \in K_m^M(k)$.

Proof. See [GS17, Proposition 7.1.1, Lemma 7.1.2]. $\square$

Remark 2.2.4. Let K be a field equipped with a discrete valuation $v\colon K \to \mathbb{Z} \cup \{\infty\}$. Let $A = \{x \in K \mid v(x) \geq 0\}$ be the associated discrete valuation ring with residue field κ. Fix a uniformizer $\pi \in A$ (i.e. $v(\pi) = 1$), then each element $x \in K^*$ can be written uniquely as a product $u\pi^{v(x)}$ for some unit $u \in A^*$. Hence check easily by bilinearity and graded-commutativity that the groups $K_n^M(K)$ are generated by symbols of the form $\{\pi, u_2, \dots, u_n\}$ and $\{u_1, \dots, u_n\}$, where the u_i are units in A.

Until further notice, we keep the notation from above, i.e. let K be a field with a discrete valuation, associated discrete valuation ring A with residue field κ, and fix an arbitrary uniformizer π.

Proposition 2.2.5. (Tame Symbol) For each $n \geq 1$ there exists a unique homomorphism

$$\partial^M\colon K_n^M(K) \to K_{n-1}^M(\kappa)$$

satisfying

$$\partial^M(\{\pi, u_2, \dots, u_n\}) = \{\overline{u}_2, \dots, \overline{u}_n\} \tag{2.2.2}$$

for *all* uniformizers π and all $(n-1)$-tuples $(u_2, \dots, u_n)$ of units in A, where $\overline{u}_i$ denotes the image of u_i in κ.

The map ∂^M is called the *tame symbol* or the *residue map* for Milnor K-theory.

Proof. The uniqueness follows from the above Remark 2.2.4 on generators of $K_n^M(K)$, once we see that a symbol $\{u_1, \dots, u_n\}$ can be written as $\{\pi u_1, u_2, \dots, u_n\} - \{\pi, u_2, \dots, u_n\}$. As both π and πu_1 are uniformizers, we have by (2.2.2)

$$\partial^M(\{u_1, \dots, u_n\}) = \{\overline{u}_2, \dots, \overline{u}_n\} - \{\overline{u}_2, \dots, \overline{u}_n\} = 0.$$

Hence the image $\partial^M(\alpha)$ for $\alpha \in K_n^M(K)$ is uniquely defined by (2.2.2). Proving that ∂^M does indeed exist is harder though, hence we refer to [GS17, Proposition 7.1.4] for details. $\square$

Example 2.2.6.

i.) The tame symbol $\partial^M\colon K_1(K) \to K_0(\kappa)$ is obviously none but the valuation map $v\colon K^* \to \mathbb{Z}$.

ii.) The tame symbol $\partial^M \colon K_2(K) \to K_1(\kappa)$ is given by the formula

$$\partial^M(\{a, b\}) = (-1)^{v(a)v(b)} \overline{a^{-v(b)} b^{v(a)}}.$$

Indeed, write $a = u_1 \pi^{v(a)}$ and $b = u_2 \pi^{v(b)}$. Then with Proposition 2.2.3 obtain

$$\{a, b\} = \{u_1, u_2\} + v(b) \underbrace{\{u_1, \pi\}}_{=-\{\pi, u_1\}} + v(a)\{\pi, u_2\} + v(a)v(b) \underbrace{\{\pi, \pi\}}_{=\{\pi, -1\}}.$$

Identifying $K_1(\kappa)$ with κ^* and applying ∂^M yields with (2.2.2)

$$\begin{aligned} \partial^M(\{a, b\}) &= -v(b)\{\overline{u}_1\} + v(a)\{\overline{u}_2\} + v(a)v(b)\{-1\} \\ &= (-1)^{v(a)v(b)} \overline{u}_1^{-v(b)} \overline{u}_2^{v(a)} \\ &= (-1)^{v(a)v(b)} \overline{a^{-v(b)} b^{v(a)}}. \end{aligned}$$

Before we introduce Gersten sequences, we are going to determine the kernel of the tame symbol and introduce the norm map.

Proposition 2.2.7. Let U_n be the subgroup of $K_n^M(K)$ generated by those symbols $\{u_1, \ldots, u_n\}$ where all the u_i are units in A. Then we have an exact sequence

$$0 \longrightarrow U_n \longrightarrow K_n^M(K) \xrightarrow{\partial^M} K_{n-1}^M(\kappa) \longrightarrow 0.$$

Proof. The tame symbol is clearly surjective by definition, and clearly the sequence is a complex. Hence we only need to show that $\ker(\partial^M) = U_n$. For this consider the subgroup $U_n^1 \subseteq K_n^M(K)$ generated by symbols $\{x_1, \ldots, x_n\}$ with $x_1 \in A^*$ satisfying $\overline{x}_1 = 1 \in \kappa$. One can show with the Steinberg relation that U_n^1 is contained in U_n (see [GS17, Lemma 7.1.8]). Consider the map

$$\begin{gathered} \psi \colon K_{n-1}^M(\kappa) \to K_n^M(K)/U_n^1 \\ \{\overline{u}_1, \ldots, \overline{u}_{n-1}\} \to \{\pi, u_1, \ldots, u_{n-1}\} \bmod U_n^1, \end{gathered}$$

where the $u_i \in A^*$ are arbitrary lifings of the $\overline{u}_i$. This map is well-defined, because if u_i and u_i' are two liftings of $\overline{u}_i$, then $u_i' = u_i + a\pi = u_i \underbrace{(1 + u_i^{-1} a\pi)}_{\in U_n^1}$ for some $a \in A$. Hence replacing some lifting by another modifies $\{\pi, u_1, \ldots, u_{n-1}\}$ by an element in U_n^1.

Now let $\beta \in \ker(\partial^M)$. Without loss of generality we can assume that β is of the form $\beta = \{\pi, u_2, \ldots, u_n\}$. Then we have $0 = (\psi \circ \partial^M)(\beta) = \beta \bmod U_n^1$, i.e. $\beta \in U_n^1 \subset U_n$. This shows the claim. $\square$

We briefly introduce the norm map, which will be necessary later for defining the coboundary maps in the Gersten complex.

Definition 2.2.8. Given an inclusion of fields $\phi\colon k \hookrightarrow K$, there is a natural map $i_{K|k}\colon K_n^M(k) \to K_n^M(K)$ induced by ϕ. Given $\alpha \in K_n^M(k)$ we often abbreviate $i_{K|k}(\alpha)$ by α_K.

Proposition 2.2.9. Let K/k be a finite field extension. For all $n \geq 0$ there is a norm morphism

$$N_{K|k}\colon K_n^M(K) \to K_n^M(k)$$

satisfying the following properties:

(1) The map $N_{K|k}\colon K_0^M(K) \to K_0^M(k)$ is given by multiplication by the degree $[K\colon k]$.

(2) The map $N_{K|k}\colon K_1^M(K) \to K_1^M(k)$ is given by the usual field norm $N_{K|k}\colon K^* \to k^*$.

(3) Given $\alpha \in K_n^M(k)$ and $\beta \in K_m^M(K)$, one has

$$N_{K|k}(\{\alpha_K, \beta\}) = \{\alpha, N_{K|k}(\beta)\}.$$

(4) Given a tower of field extensions $K'/K/k$, one has

$$N_{K'|K} \circ N_{K|k} = N_{K'|k}.$$

Proof. We will not go into the construction of the norm map, but refer to chapter 7.3 of [GS17]. $\square$

We can now finally describe our main object of study in this chapter, the Gersten complex.

Definition 2.2.10. Let X be a smooth n-dimensional variety over a field K. For every $p \geq 0$ we have a complex of abelian groups

$$S_p(X)\colon \bigoplus_{x\in X^{(0)}} K_p^M(k(x)) \xrightarrow{d} \bigoplus_{x\in X^{(1)}} K_{p-1}^M(k(x)) \xrightarrow{d} \cdots \xrightarrow{d} \bigoplus_{x\in X^{(p)}} K_0^M(k(x)) \longrightarrow 0,$$

called the *Gersten complex.* Here $k(x)$ denotes the residue field $\mathcal{O}_{X,x}/\mathfrak{m}_{X,x}$. The coboundary maps

$$d\colon \bigoplus_{x\in X^{(q)}} K_{p-q}^M(k(x)) \to \bigoplus_{x\in X^{(q+1)}} K_{p-q-1}^M(k(x))$$

are given as follows:

In the following construction keep in mind that due to the surjectivity of the induced map on stalks given by a closed immersion, for any closed subvariety $V := \overline{\{y\}} \subseteq X$ of any variety X we have a field isomorphism

$$K(V) = \mathcal{O}_{V,y} \cong \mathcal{O}_{X,y}/\mathfrak{m}_{X,y} = k(y). \tag{2.2.3}$$

Now to construct d, consider a point $x \in X^{(q)}$ and let $Z_x = \overline{\{x\}}$ be its corresponding closed subvariety in X. Each point y of codimension 1 in Z_x (with $V := \overline{\{y\}}$ the corresponding variety) is a point in $X^{(q+1)}$. As in Remark 2.1.3, if $\varphi\colon \tilde{Z}_x \to Z_x$ is the normalization of Z_x, the fiber $\varphi^{-1}(\{y\})$ is finite consisting of points $\tilde{y}_1, \dots, \tilde{y}_r$ (with corresponding varieties $V_i := \overline{\{\tilde{y}_i\}}$, hence $\varphi(V_i) = V$ as φ is finite and thus closed). As $\tilde{Z}_x$ is normal, for all i the local ring $(\mathcal{O}_{\tilde{Z}_x,\tilde{y}_i}, \mathfrak{m}_{\tilde{Z}_x,\tilde{y}_i})$ is a discrete valuation ring [GW10, Proposition 10.37], hence defines as usual via the order of vanishing a discrete valuation on the function field $\operatorname{Frac}(\mathcal{O}_{\tilde{Z}_x,\tilde{y}_i}) = K(\tilde{Z}_x) = K(Z_x) = k(x)$.

Now if

$$\partial^M_{\tilde{y}_i}\colon K^M_{p-q}(\underbrace{k(x)}_{=K(\tilde{Z}_x)}) \to K^M_{p-q-1}(\underbrace{k(\tilde{y}_i)}_{=\mathcal{O}_{\tilde{Z}_x,\tilde{y}_i}/\mathfrak{m}_{\tilde{Z}_x,\tilde{y}_i}})$$

is the tame symbol, we may define a map

$$\partial^x_y\colon K^M_{p-q}(k(x)) \to K^M_{p-q-1}(k(y))$$

by setting

$$\partial^x_y := \sum_{i=1}^r N_{\underbrace{k(\tilde{y}_i)|k(y)}_{=K(V_i)|K(V)}} \circ \partial^M_{\tilde{y}_i},$$

where $N_{k(\tilde{y}_i)|k(y)}$ is the norm morphism from Proposition 2.2.9.

Since $\tilde{Z}_x$ is a Noetherian integral scheme, any $f \in k(x)^* = K(\tilde{Z}_x)^*$ satisfies $f \in \mathcal{O}^*_{\tilde{Z}_x,\tilde{y}}$ for almost all codimension one points $\tilde{y} \in \tilde{Z}_x$ (see Definition 2.1.2), in particular the associated valuation $\operatorname{ord}_{\overline{\{\tilde{y}\}}}(f)$ is trivial for all but finitely many $\tilde{y}$ (note that here $\tilde{y}$ runs over *all* codimension 1 points in $\tilde{Z}_x$).

Hence by Remark 2.2.4, for a fixed $\alpha \in K^M_{p-q}(k(x))$, the image $\partial^x_y(\alpha)$ under the tame symbol is trivial for all but finitely many y, and we can consider the sum

$$\partial_x := \left(\sum_{y \in Z_x^{(1)}} \partial^x_y\right) : K^M_{p-q}(k(x)) \to \bigoplus_{x \in X^{(q+1)}} K^M_{p-q-1}(k(x)),$$

and finally obtain d as the direct sum of the maps ∂_x for all $x \in X^{(q)}$.

Remark 2.2.11. We succinctly mentioned that the sequence $S_p(X)$ is a complex. This result is due to Kato, and is far from trivial. For details of the proof see e.g. [GS17, Theorem 8.1.2].

Remark 2.2.12. (Connection to Chow groups) The Chow group $\mathrm{CH}^p(X)$ arises naturally in the complex $S_p(X)$. To see this, we look at the last map

$$\bigoplus_{X^{(p-1)}} K_1^M(k(x)) \xrightarrow{d} \bigoplus_{X^{(p)}} K_0^M(k(x))$$

which by Definition 2.2.1 is nothing but

$$\bigoplus_{X^{(p-1)}} k(x)^* \xrightarrow{d} \bigoplus_{X^{(p)}} \mathbb{Z}.$$

Keeping the notation from the construction of d above, for an $x \in X^{(p-1)}$ the tame symbol $\partial^M_{\tilde{y}_i} : k(x)^* \to \mathbb{Z}$ is exactly the induced discrete valuation (see Example 2.2.6). Furthermore the norm map $N_{k(\tilde{y}_i)|k(y)}$ is given by multiplication with the degree $[k(\tilde{y}_i) : k(y)]$ by Proposition 2.2.9 (1). Hence this yields for $y \in X^{(p)}$ of codimension 1 in Z_x with $V = \overline{\{y\}}$ as above, that

$$\partial^x_y(f) = \sum_{i=1}^{r} \mathrm{ord}_{V_i}(f)[K(V_i) : K(V)] \overset{(2.1.2)}{=} \mathrm{ord}_V(f)$$

for $f \in k(x)^*$ and thus

$$\partial_x(f) = \mathrm{div}(f)$$

for $f \in k(x)^*$.

Hence

$$\mathrm{CH}^p(X) \cong \mathrm{coker}\left(\bigoplus_{x \in X^{(p-1)}} K_1^M(k(x)) \xrightarrow{d} \bigoplus_{x \in X^{(p)}} K_0^M(k(x)) \right).$$

This isomorphism is the starting point for Bloch's formula.

2.3 Bloch's formula

Remark 2.3.1. In the following we write $\mathrm{codim}_Y X$ for the codimension of an arbitrary subset Y of a scheme X, i.e.

$$\mathrm{codim}_Y X = \inf_{y \in Y} \dim \mathcal{O}_{X,y}.$$

Definition 2.3.2. Let $(X, \mathcal{O}_X)$ be a ringed space. The assignment

$$U \mapsto K_q^M(\mathcal{O}_X(U)) \otimes_{\mathbb{Z}} \mathbb{Q} \tag{2.3.1}$$

for an open subset $U \subseteq X$ defines a presheaf (the restriction maps of $\mathcal{O}_X$ clearly preserve the Steinberg relation).

Define the *q-th sheaf of rational Milnor K-Theory* $\mathscr{K}_X^q$ for $(X, \mathcal{O}_X)$ as the sheafification of (2.3.1).

In the following let X always be a smooth variety over an algebraically closed field K with generic point η. For $x \in X$, denote by

$$i_x \colon \operatorname{Spec}(k(x)) \to X$$

its induced morphism of K-schemes. Note that the presheaf (2.3.1) is clearly a sheaf on the one-point set $\operatorname{Spec}(k(x))$ for every integer $q \geq 0$. In particular we can consider the sheaf $i_{x*}K_q^M(k(x)) \otimes_{\mathbb{Z}} \mathbb{Q}$ on X where an open $U \subseteq X$ is mapped to

$$(i_{x*}K_q^M(k(x)) \otimes_{\mathbb{Z}} \mathbb{Q})(U) = \begin{cases} 0 & \text{if } x \notin U \\ K_q^M(k(x)) \otimes_{\mathbb{Z}} \mathbb{Q} & \text{if } x \in U. \end{cases} \tag{2.3.2}$$

Passing to the direct sum sheaf we further consider for every $p, q \geq 0$ the sheaf

$$\mathscr{K}_X^{p,q} := \bigoplus_{x \in X^{(q)}} i_{x*}K_{p-q}^M(k(x)) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

One important thing to note is that as X has – as a variety – an underlying Noetherian topological space, every open subset is quasi-compact. In particular we see that the direct sum sheaf does not require sheafification and for an open $U \subseteq X$ we obtain the sections

$$\mathscr{K}_X^{p,q}(U) = \bigoplus_{x \in X^{(q)}} (i_{x*}K_{p-q}^M(k(x)) \otimes_{\mathbb{Z}} \mathbb{Q})(U) \cong \bigoplus_{x \in X^{(q)} \cap U} K_{p-q}^M(k(x)) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Remark 2.3.3.

i.) The description of sections (2.3.2) shows immediately that the restriction maps of $\mathscr{K}_X^{p,q}$ are surjective, hence the sheaves $\mathscr{K}_X^{p,q}$ are flasque.

ii.) Let $U \subseteq X$ be open and $x \in U$. As

$$\operatorname{codim}_X(\overline{\{x\}}) = \dim \mathcal{O}_{X,x} = \dim \mathcal{O}_{U,x} = \operatorname{codim}_U(\overline{\{x\}})$$

we have for any $q \geq 0$ that $X^{(q)} \cap U = U^{(q)}$. Hence for an open $V \subseteq U$ obtain

$$\begin{aligned}\mathscr{K}_X^{p,q}|_U(V) = \mathscr{K}_X^{p,q}(U \cap V) &\cong \bigoplus_{x \in X^{(q)} \cap U \cap V} K^M_{p-q}(k(x)) \otimes \mathbb{Q} \\ &\cong \bigoplus_{x \in U^{(q)} \cap V} K^M_{p-q}(k(x)) \otimes \mathbb{Q} \cong \mathscr{K}_U^{p,q}(V),\end{aligned}$$

and thus a sheaf isomorphism $\mathscr{K}_X^{p,q}|_U \cong \mathscr{K}_U^{p,q}$.

Lemma 2.3.4. Let $Z \subseteq X$ be a closed subvariety of X of codimension p and $U := X \setminus Z$ its open complement. Then for all $q \leq p-1$ we have isomorphisms

$$\mathscr{K}_X^{p,q}(X) \cong \mathscr{K}_U^{p,q}(U).$$

Proof. Let $q \leq p-1$. As seen above, we have $\mathscr{K}_U^{p,q}(U) = \mathscr{K}_X^{p,q}(U)$ and it suffices to show that $X^{(q)} = X^{(q)} \cap U$. Initially note that for any variety X and closed subset Y we have (see e.g. [GW10, Proposition 5.30])

$$\dim Y + \mathrm{codim}_X Y = \dim X.$$

Now suppose there is $x \in X^{(q)}$ with $V := \overline{\{x\}}$ such that $x \notin U$. Then $x \in Z$, i.e. $V \subseteq Z$. As Z is also a variety, obtain

$$\underbrace{\dim Z}_{=\dim V + \mathrm{codim}_Z V} + \mathrm{codim}_X Z = \dim X = \dim V + \mathrm{codim}_X V,$$

thus

$$\mathrm{codim}_Z V + p = \mathrm{codim}_Z V + \mathrm{codim}_X Z = \mathrm{codim}_X V = q, \tag{2.3.3}$$

in particular $p \leq q$. For $q \leq p-1$ this yields a contradiction. $\square$

Remark 2.3.5. Note that the coboundary maps d in the Gersten complex 2.2.10 commute with restriction, i.e. for open subsets $V \subseteq U$ of X the following diagram commutes

$$\begin{array}{ccc}\bigoplus_{x \in X^{(q)} \cap U} K^M_{p-q}(k(x)) & \xrightarrow{d} & \bigoplus_{x \in X^{(q+1)} \cap U} K^M_{p-q-1}(k(x)) \\ \downarrow & & \downarrow \\ \bigoplus_{x \in X^{(q)} \cap V} K^M_{p-q}(k(x)) & \xrightarrow{d} & \bigoplus_{x \in X^{(q+1)} \cap V} K^M_{p-q-1}(k(x)).\end{array}$$

Indeed any point $y \in X^{(q+1)} \cap V$ that is of codimension 1 for some $x \in X^{(q)} \cap U$ implies already that $x \in V$ (otherwise $y \in \overline{\{x\}} \subseteq X \setminus V$).

In particular we get a sheaf homomorphism

$$d\colon \mathscr{K}_X^{p,q} \to \mathscr{K}_X^{p,q+1}.$$

Furthermore there is a morphism of sheaves $\epsilon\colon \mathscr{K}_X^p \to \mathscr{K}_X^{p,0}$ induced by the presheaf morphism which is given for an open $U \subseteq X$ on the groups of Milnor K-Theory by

$$K_p^M(\mathcal{O}_X(U)) \to \bigoplus_{x \in X^{(0)}} K_p^M(k(x)) \cong K_p^M(k(\eta)),$$
$$\{f_1, \ldots, f_p\} \mapsto \{f_{1,\eta}, \ldots, f_{p,\eta}\}.$$

Here $f_i \in \mathcal{O}_X(U)^*$ and η denotes as usual the generic point of X. We have the relation $d \circ \epsilon = 0$. Indeed, as X is smooth, for any open $U \subseteq X$ the map

$$d\colon \mathscr{K}_X^{p,0}(U) \cong K_p^M(k(\eta)) \otimes \mathbb{Q} \longrightarrow \bigoplus_{x \in X^{(1)} \cap U} K_{p-1}^M(k(x)) \otimes \mathbb{Q} \cong \mathscr{K}_X^{p,1}(U)$$

is given exactly by the tame symbol for any summand $x \in X^{(1)} \cap U$. By description of the kernel in Proposition 2.2.7, we obtain $d \circ \epsilon = 0$ as any $\{f_1, \ldots, f_p\} \in K_p^M(\mathcal{O}_X(U))$ satisfies $f_{i,\eta} \in \mathcal{O}_{X,x}^*$ for all i and any $x \in X^{(1)} \cap U$.

As a consequence, we obtain a complex of flasque sheaves

$$0 \longrightarrow \mathscr{K}_X^p \xrightarrow{\epsilon} \mathscr{K}_X^{p,0} \xrightarrow{d} \mathscr{K}_X^{p,1} \xrightarrow{d} \mathscr{K}_X^{p,2} \xrightarrow{d} \cdots \xrightarrow{d} \mathscr{K}_X^{p,p} \longrightarrow 0.$$

Theorem 2.3.6. The complex of sheaves

$$0 \longrightarrow \mathscr{K}_X^p \xrightarrow{\epsilon} \mathscr{K}_X^{p,0} \xrightarrow{d} \mathscr{K}_X^{p,1} \xrightarrow{d} \mathscr{K}_X^{p,2} \xrightarrow{d} \cdots \xrightarrow{d} \mathscr{K}_X^{p,p} \longrightarrow 0 \tag{2.3.4}$$

is exact.

Proof. See e.g. [Sou85, Théorème 5]. □

Remark 2.3.7. The complex $\mathscr{K}_X^{p,\bullet}$ is a flasque resolution of $\mathscr{K}_X^p$. Hence it can be used to compute the sheaf cohomology groups of $\mathscr{K}_X^p$ and we obtain

$$\mathrm{H}^q(X, \mathscr{K}_X^p) \cong \frac{\ker(d\colon \mathscr{K}_X^{p,q}(X) \to \mathscr{K}_X^{p,q+1}(X))}{\mathrm{im}(d\colon \mathscr{K}_X^{p,q-1}(X) \to \mathscr{K}_X^{p,q}(X))}.$$

Lemma 2.3.8. Let X be a smooth variety over a field K, and $Z = \overline{\{z\}} \subseteq X$ a closed subvariety of codimension p. Let $U = X \setminus Z$ be the open complement of Z. Then the following holds:

i.) (Bloch's formula) For every $p \geq 0$, there is a canonical isomorphism

$$\mathrm{H}^p(X, \mathscr{K}_X^p) \cong \mathrm{CH}^p(X)_{\mathbb{Q}} := \mathrm{CH}^p(X) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

ii.) For $q > p \geq 0$, we have $H^q(X, \mathscr{K}_X^p) = 0$.

iii.) For $q \neq p$ we have

$$\mathrm{H}_Z^q(X, \mathscr{K}_X^p) = 0,$$

and

$$\mathrm{H}_Z^p(X, \mathscr{K}_X^p) \cong \mathbb{Q}.$$

Here $\mathrm{H}_Z^q(X, \mathscr{K}_X^p)$ denotes the cohomology group with support in Z (see Appendix A).

Proof. Both i.) and ii.) follow from the flasque resolution (2.3.4) of $\mathscr{K}_X^p$ and its connection to Chow groups as explained in Remark 2.2.12. We show iii.). Note that as the $\mathscr{K}_X^{p,q}$ are flasque sheaves, we have by A.5 iii.) a short exact sequence

$$0 \to \Gamma_Z(X, \mathscr{K}_X^{p,q}) \to \mathscr{K}_X^{p,q}(X) \to \mathscr{K}_U^{p,q}(U) \to 0.$$

Here the map $\mathscr{K}_X^{p,q}(X) \to \mathscr{K}_U^{p,q}(U)$ is an isomorphism by Lemma 2.3.4 for $q \leq p-1$, hence

$$\Gamma_Z(X, \mathscr{K}_X^{p,q}) = 0$$

and in particular $\mathrm{H}_Z^q(X, \mathscr{K}_X^p) = 0$ for $q \leq p-1$.

To show the last part, we first want to see how generally the global sections with support $\Gamma_Z(X, \mathscr{K}_X^{p,q})$ look like. For this let $\alpha = (\alpha_x)_{x \in X^{(q)}} \in \bigoplus_{x \in X^{(q)}} K_{p-q}^M(k(x)) \otimes \mathbb{Q}$. Then with the restriction maps see easily that $\mathrm{supp}(\alpha) \subseteq Z$ if and only if $\alpha_x = 0$ for all $x \in U$. Hence

$$\Gamma_Z(X, \mathscr{K}_X^{p,q}) \cong \bigoplus_{x \in X^{(q)} \cap Z} K_{p-q}^M(k(x)) \otimes \mathbb{Q}.$$

Any closed subset of Z of codimension 0 must already be equal to the irreducible Z, hence $X^{(p)} \cap Z = \{z\}$ by (2.3.3). As $\Gamma_Z(X, \mathscr{K}_X^{p,q}) = 0$ for $q > p$ and $q < p$, we have

$$\mathrm{H}_Z^p(X, \mathscr{K}_X^p) = \Gamma_Z(X, \mathscr{K}_X^{p,p}) = \bigoplus_{x \in X^{(p)} \cap Z} \mathbb{Z} \otimes \mathbb{Q} \cong \mathbb{Q}. \tag{2.3.5}$$

□

In the following we will denote by

$$\mathrm{cl}_{\mathrm{univ}} \colon \mathrm{CH}^p(X)_{\mathbb{Q}} \to \mathrm{H}^p(X, \mathscr{K}_X^p)$$

the map obtained in Bloch's formula in Lemma 2.3.8 i.).

Remark 2.3.9. In the long exact sequence of cohomology with supports (see Corollary A.6), the map

$$\mathrm{H}^p_Z(X, \mathscr{K}^p_X) \to \mathrm{H}^p(X, \mathscr{K}^p_X) \cong \mathrm{CH}^p(X)_{\mathbb{Q}}$$

comes from the canonical morphism

$$\begin{array}{ccc}
\mathrm{H}^p_Z(X, \mathscr{K}^p_X) & \cong & \mathbb{Z} \otimes \mathbb{Q} \\
\| & & \wr\| \\
\Gamma_Z(X, \mathscr{K}^{p,p}_X) & = & \bigoplus_{x \in X^{(p)} \cap Z} \mathbb{Z} \otimes \mathbb{Q} \\
\downarrow & & \downarrow \\
\Gamma(X, \mathscr{K}^{p,p}_X) & = & \bigoplus_{x \in X^{(p)}} \mathbb{Z} \otimes \mathbb{Q}.
\end{array}$$

In particular the image of 1 under the map $\mathrm{H}^p_Z(X, \mathscr{K}^p_X) \to \mathrm{H}^p(X, \mathscr{K}^p_X) \cong \mathrm{CH}^p(X)_{\mathbb{Q}}$ is exactly the class $[Z]$ of the closed subvariety Z.

2.4 A special description of $\mathrm{cl}_{\mathrm{univ}}$

In this section we want to give a description by regular sequences of the image $\mathrm{cl}_{\mathrm{univ}}([Z])$ for a smooth closed subvariety Z of codimension p in our smooth variety X. Note that as we are working over an algebraically closed field K, the term 'smooth' is equivalent to 'regular'. This special description plays a crucial role in Liu's proof of Theorem 4.2.5. A main part of this section explicitly follows the canonical morphism from Čech cohomology to sheaf cohomology (see Remark B.7); we refer to Appendix B for a detailed discussion.

Construction 2.4.1. Let Z be a smooth closed subvariety of X of codimension p. Choose a finite affine open covering U_α of X and a regular sequence $f_{\alpha 1}, \ldots, f_{\alpha p} \in \mathcal{O}_X(U_\alpha)$ such that $Z \cap U_\alpha$ is defined by the ideal $(f_{\alpha 1}, \ldots, f_{\alpha p})$. This choice is possible by [Ful98, B.7.2], as both Z and X are smooth. Write $U_{\alpha i}$ for the nonvanishing locus $D(f_{\alpha i}) \subseteq U_\alpha$ of $f_{\alpha i}$. Then $\{U_{\alpha i} \mid 1 \leq i \leq p\}$ forms an open covering of $U_\alpha \setminus Z$, which we will denote by $\mathfrak{U}_\alpha$.

As $\mathfrak{U}_\alpha$ consists of only p sets, the element $\{f_{\alpha 1}, \ldots, f_{\alpha p}\} \in K^M_p(\mathcal{O}_X(\bigcap_{i=1}^p U_{\alpha i}))$ forms a Čech cocycle $\mu \in C^{p-1}(\mathfrak{U}_\alpha, \mathscr{K}^p_{U_\alpha \setminus Z})$. We obtain an element $c(Z)_\alpha \in \mathrm{H}^p(U_\alpha \setminus Z, \mathscr{K}^p_{U_\alpha \setminus Z})$ as the image of μ under the composition

$$\check{\mathrm{H}}^{p-1}(\mathfrak{U}_\alpha, \mathscr{K}^p_{U_\alpha \setminus Z}) \to \mathrm{H}^{p-1}(U_\alpha \setminus Z, \mathscr{K}^p_{U_\alpha \setminus Z}) \to \mathrm{H}^p_{Z \cap U_\alpha}(U_\alpha, \mathscr{K}^p_{U_\alpha}). \tag{2.4.1}$$

Here the first map is the canonical morphism from Čech cohomology to sheaf cohomology in Remark B.7, and the second map is the connecting morphism in the long exact cohomology sequence with supports (see Corollary A.6).

Our main goal in this section is to prove the following proposition.

Proposition 2.4.2. Assume that $Z \cap U_\alpha$ is nonempty. Then the image of $c(Z)_\alpha \in \mathrm{H}^p_{Z\cap U_\alpha}(U_\alpha, \mathscr{K}^p_{U_\alpha})$ under the isomorphism $\mathrm{H}^p_{Z\cap U_\alpha}(U_\alpha, \mathscr{K}^p_{U_\alpha}) \cong \mathbb{Q}$ in Lemma 2.3.8 iii.) equals 1. In particular, the above construction does not depend on the choice of the regular sequence $f_{\alpha 1}, \ldots, f_{\alpha p}$.

Before we prove the proposition, we make the following observations which are crucial ingredients for the proof.

Lemma 2.4.3. Let $X = \mathrm{Spec}(A)$ be a smooth affine variety, and $f_1, \ldots, f_p \in A \setminus \{0\}$. Set $U := \bigcap_{i=2}^p D(f_i)$. We consider a point $x \in U$ with closed subvarieties $Z = \overline{\{x\}}$ and $V := \overline{\{y\}}$ for a point $y \in Z \cap U$ with $\mathrm{codim}_Z V = 1$. Furthermore let

$$\partial^x_y \colon K^M_p(k(x)) \to K^M_{p-1}(k(y))$$

be the morphism as in Definition 2.2.10. Note that we have

$$\mathrm{Frac}(\mathcal{O}_{Z,y}) = K(Z) = \mathcal{O}_{Z,x} \cong \mathcal{O}_{X,x}/\mathfrak{m}_{X,x} = k(x).$$

Then the following holds:

i.) The germ $f_{i,x} \in k(x)$ lies in $\mathcal{O}^*_{Z,y}$ for $2 \leq i \leq p$.

ii.) We have $\partial^x_y(\{f_{1,x}, \ldots, f_{p,x}\}) = \mathrm{ord}_V(f_{1,x}) \cdot \{\widetilde{f_{2,x}}, \ldots, \widetilde{f_{p,x}}\}$.

iii.) If $y \in D(f_1)$, then $\partial^x_y(\{f_{1,x}, \ldots, f_{p,x}\}) = 0$.

Here $\widetilde{f_{i,x}}$ denotes the residue class of $f_{i,x} \in \mathcal{O}^*_{Z,y}$ taken in $k(y) = \mathcal{O}_{Z,y}/\mathfrak{m}_{Z,y}$.

Proof. Statement i.) is clear, as y lies in the non-vanishing locus of $f_2, \ldots, f_p$. Also iii.) follows directly from ii.), because if $y \in D(f_1)$, then $\mathrm{ord}_V(f_{1,x}) = 0$.

To show ii.), we denote by $\tilde{Z}$ the normalization of Z, which is given by the affine scheme corresponding to the integral closure of $\mathcal{O}_Z(Z)$, and denote by $y_1, \ldots, y_r \in \tilde{Z}$ the points lying above y. As $f_{i,x} \in \mathcal{O}^*_{Z,y}$, we also have $f_{i,x} \in \mathcal{O}^*_{\tilde{Z},y_j}$ for all $j = 1, \ldots, r$ and $i \geq 2$, via the inclusion $\mathcal{O}_{Z,y} \hookrightarrow \mathcal{O}_{\tilde{Z},y_j}$.

Using the notation as in Definition 2.2.10, by the property of the tame symbol, we have

$$\partial_y^x(\{f_{1,x},\dots,f_{p,x}\}) = \sum_{j=1}^r N_{k(y_j)|k(y)} \circ \partial_{y_j}^M(\{f_{1,x},\dots,f_{p,x}\}) =$$
$$\sum_{j=1}^r \operatorname{ord}_{\overline{\{y_j\}}}(f_{1,x}) \cdot N_{k(y_j)|k(y)}(\{\widetilde{f_{2,x}}^j,\dots,\widetilde{f_{p,x}}^j\}),$$

where $\operatorname{ord}_{\overline{\{y_j\}}}(f_{1,x})$ denotes the valuation of $f_{1,x} \in K(Z) = K(\tilde{Z}) = \operatorname{Frac}(\mathcal{O}_{\tilde{Z},y_j})$ with respect to y_j, and $\widetilde{f_{i,x}}^j$ is the residue class of $f_{i,x} \in \mathcal{O}^*_{Z,y} \hookrightarrow \mathcal{O}^*_{\tilde{Z},y_j}$ taken in $k(y_j) = \mathcal{O}_{\tilde{Z},y_j}/\mathfrak{m}_{\tilde{Z},y_j}$.

Note that by property (1) and (3) of the norm map 2.2.9, and as already $\widetilde{f_{i,x}}^j \in k(y) = \mathcal{O}_{Z,y}/\mathfrak{m}_{Z,y}$, the above equation simplifies to

$$\begin{aligned}\partial_y^x(\{f_{1,x},\dots,f_{p,x}\}) &= \sum_{j=1}^r \operatorname{ord}_{\overline{\{y_j\}}}(f_{1,x}) \cdot [k(\tilde{y}_j):k(y)] \cdot \{\widetilde{f_{2,x}},\dots,\widetilde{f_{p,x}}\} \\ &= \operatorname{ord}_V(f_{1,x}) \cdot \{\widetilde{f_{2,x}},\dots,\widetilde{f_{p,x}}\},\end{aligned}$$

where the second equation comes from the characterization of the order of vanishing as in Remark 2.1.3. □

Lemma 2.4.4. Let $X = \operatorname{Spec}(A)$ be a smooth affine variety, and $f_1,\dots,f_p \in A$ a regular sequence, such that $Z := V(f_1,\dots,f_p) = \overline{\{z\}}$ is a smooth closed subvariety of codimension p. Let $1 \le n \le p$ be arbitrary.

i.) The closed subscheme $V(f_1,\dots,f_n)$ is of pure codimension n in X, i.e. every irreducible component of $V(f_1,\dots,f_n)$ is of codimension n in X. Let $Y_1,\dots,Y_r$ be the finitely many irreducible components of $Y := V(f_1,\dots,f_{n-1})$. Then any nonempty $W_i := V(f_n) \cap Y_i$ is of pure codimension 1 over Y_i.

ii.) The closed subscheme $V(f_1,\dots,f_n)$ is smooth along Z, and Z is contained in exactly one irreducible component of $V(f_1,\dots,f_n)$.

Proof. Statement i.) follows from an inductive application of a geometric version of Krull's Hauptidealsatz: As f_1 is neither 0 nor a unit, by [GW10, Theorem 5.32] the closed subscheme $\emptyset \subsetneq V(f_1) \subsetneq X$ is of pure codimension 1 in X. For the inductive step, let the claim be true for $Y := V(f_1,\dots,f_{n-1})$ with irreducible components $Y_1,\dots,Y_r \subseteq Y$ of codimension $n-1$

in X. We regard the Y_i's with their unique integral subscheme structure and obtain

$$V(f_1,\dots,f_n) = V(f_n)\cap Y = \bigcup_{i=1}^{r}(V(f_n)\cap Y_i).$$

Let $W_i := V(f_n)\cap Y_i$ and suppose $W_i \neq \emptyset$. Write $B := A/(f_1,\dots,f_{n-1})$ and let $\mathfrak{p}\in \mathrm{Spec}(B)$ be the minimal prime ideal corresponding to Y_i, in particular, as Y_i is integral, we can assume $Y_i = \mathrm{Spec}(B/\mathfrak{p})$. As $W_i \neq \emptyset$, we have that f_n is not a unit in $B/\mathfrak{p}$, and also f_n is not zero in $B/\mathfrak{p}$, because otherwise f_n would be a zero divisor on $B = A/(f_1,\dots,f_{n-1})$ (recall that in Noetherian rings, minimal prime ideals consist of zero divisors), in contradiction to $f_1,\dots,f_n$ regular. Hence we can apply [GW10, Theorem 5.32] again and obtain that W_i is of pure codimension 1 in Y_i. Together with the induction hypothesis this yields that $V(f_1,\dots,f_n)$ is of pure codimension n in X. This shows i.).

For ii.) we consider ideals $\mathfrak{a} := (f_1,\dots,f_n)$, $\mathfrak{q} := (f_1,\dots,f_p)\subseteq A$ and $\mathfrak{p}\in Z = V(\mathfrak{q})\subseteq \mathrm{Spec}(A)$. In particular we have a chain of inclusion $\mathfrak{a}\subseteq\mathfrak{q}\subseteq\mathfrak{p}$. We need to show that the Noetherian local ring $B := (A/\mathfrak{a})_{\mathfrak{p}} \cong A_{\mathfrak{p}}/\mathfrak{a}A_{\mathfrak{p}}$ is regular. As localization and taking quotients commute, see easily that (by some abuse of notation):

- We have $B_{\mathfrak{q}} \cong A_{\mathfrak{q}}/\mathfrak{a}A_{\mathfrak{q}}$, which is regular of dimension $p-n$ and $\mathfrak{q}\in\mathrm{Spec}(B)$ is generated by $p-n$ elements. For this see Lemma 2.4.6 below and observe that regular sequences stay regular in localizations.
- Also $B/\mathfrak{q}\cong (A/\mathfrak{q})_{\mathfrak{p}}$ is regular, as $Z\cong \mathrm{Spec}(A/\mathfrak{q})$ is smooth at $\mathfrak{p}$.

Hence B is regular by [Liu06, Lemma 4.2.22].

It remains to show that $Z = \overline{\{z\}}$ is contained in exactly one irreducible component of Y. As Y is smooth along Z, the local ring $\mathcal{O}_{Y,z}$ is regular, hence a domain. The minimal primes of $\mathcal{O}_{Y,z}$ are in canonical bijection with the irreducible components of Y containing z. As $\mathcal{O}_{Y,z}$ is a domain though, there is exactly one. $\square$

Definition 2.4.5. Let $(A,\mathfrak{m})$ be a Noetherian regular local ring of dimension d. Recall that any system of generators of $\mathfrak{m}$ with d elements is called a *coordinate system* or *system of parameters* for A.

Lemma 2.4.6. Let $(A,\mathfrak{m})$ be a Noetherian regular local ring of dimension p, and let $f_1,\dots,f_p$ be a coordinate system for A. Then the ideal $\mathfrak{q} := (f_1,\dots,f_i)$ for any $1\le i\le p$ is prime, and the local domain $A/\mathfrak{q}$ is regular with coordinate system $f_{i+1},\dots,f_p$.

Proof. The statement is a direct consequence of [Liu06, Corollary 4.2.15]: The local ring $A/\mathfrak{q}$ is regular of dimension $p-i$, hence a domain and thus $\mathfrak{q}$ is prime. The maximal ideal

$\mathfrak{m}/\mathfrak{q}$ is generated by $f_{i+1} + \mathfrak{q}, \ldots, f_p + \mathfrak{q}$, so these elements form a coordinate system for $A/\mathfrak{q}$. $\square$

Remark 2.4.7. We consider again the situation where $X = \mathrm{Spec}(A)$ is a smooth affine variety with generic point η, and $f_1, \ldots, f_p \in A$ is a regular sequence, such that $Y_p := V(f_1, \ldots, f_p) = \overline{\{y_p\}}$ is a nonempty smooth closed subvariety of codimension p. By Lemma 2.4.4, we get a unique chain of inclusion

$$X =: Y_0 \supsetneq Y_1 \supsetneq \cdots \supsetneq Y_{p-1} \supsetneq Y_p = V(f_1, \ldots, f_p),$$

where $Y_i = \overline{\{y_i\}}$ is an irreducible component of $V(f_1, \ldots, f_i)$ for $1 \leq i \leq p$, and Y_{i+1} is of codimension 1 in Y_i. Note that as always we consider Y_i with its integral subscheme structure.

Hence if we consider any f_{i+1} as an element in the function field $K(Y_i) = \mathcal{O}_{Y_i, y_i}$, then $f_{i+1} \in K(Y_i)^*$, as $f_1, \ldots, f_p$ form a regular sequence. Note again that also $K(Y_i) = \mathrm{Frac}(\mathcal{O}_{Y_i, y_{i+1}})$. Then we obtain

$$\mathrm{ord}_{Y_{i+1}}(f_{i+1}) = 1.$$

To see this equality, consider the stalk $\mathcal{O}_{X,\mathfrak{p}} = A_\mathfrak{p}$, where $\mathfrak{p} = (f_1, \ldots, f_p) \in \mathrm{Spec}(A)$ (recall that Y_p is integral). As Y_p is smooth of codimension p and X is smooth, the local domain $A_\mathfrak{p}$ is regular of dimension p. Then $f_1, \ldots, f_p$ form a coordinate system for $A_\mathfrak{p}$, in particular the sequence stays regular (see [Liu06, Example 6.3.2]). By Lemma 2.4.6, the ideal $\mathfrak{q} = (f_1, \ldots, f_i) \subseteq A_\mathfrak{p}$ is prime and the local domain $A_\mathfrak{p}/\mathfrak{q}$ is regular of dimension $p - i$. We note that the prime ideal $\mathfrak{q}'$ in A corresponding to $\mathfrak{q} = \mathfrak{q}' A_\mathfrak{p}$ is clearly the unique minimal prime over $(f_1, \ldots, f_i) \subseteq A$ satisfying $(f_1, \ldots, f_i) \subseteq \mathfrak{q}' \subseteq \mathfrak{p}$. In particular we can assume that $Y_i = \mathrm{Spec}(A/\mathfrak{q}')$. We now consider the minimal prime $\mathfrak{a}$ over $f_{i+1} + \mathfrak{q}' \in A/\mathfrak{q}'$ determining Y_{i+1} with corresponding prime ideal $\mathfrak{a}' \in \mathrm{Spec}(A)$. This satisfies $(f_{i+1}, \mathfrak{q}') \subseteq \mathfrak{a}' \subseteq \mathfrak{p}$. Hence there is a prime ideal in $A_\mathfrak{p}/\mathfrak{q}$ corresponding to $\mathfrak{a}'$. By usual commutative algebra (and some abuse of notation) we have the canonical isomorphisms

$$\mathcal{O}_{Y_i, y_{i+1}} = (A/\mathfrak{q}')_\mathfrak{a} \cong (A_\mathfrak{p})_{\mathfrak{a}'}/\mathfrak{q}'(A_\mathfrak{p})_{\mathfrak{a}'} \cong (A_\mathfrak{p}/\mathfrak{q})_{\mathfrak{a}'},$$

which is a regular local domain as a localization of a regular local domain (recall that $A_\mathfrak{p}/\mathfrak{q}$ is regular by Lemma 2.4.6), and thus a discrete valuation ring. Again by Lemma 2.4.6, the maximal ideal $\mathfrak{a}'(A_\mathfrak{p}/\mathfrak{q})_{\mathfrak{a}'}$ is generated by $f_{i+1} + \mathfrak{q}$, so we get

$$\mathrm{ord}_{Y_{i+1}}(f_{i+1}) = 1$$

as intended.

Proof of Proposition 2.4.2. To ease notation, we suppress the subscript α and write $\mathfrak{U}$ for the ordered covering $\{U_1, \dots, U_p\}$ of $U \setminus Z$. For an ordered set $I \subseteq \{1, \dots, p\}$, put $U_I := \cap_{i \in I} U_i$. To even further ease and abuse notation, we often simply write $\mathscr{K}_U^{p,q}$ for the sheaf $\mathscr{K}_U^{p,q}|_{U \setminus Z} \cong \mathscr{K}_{U \setminus Z}^{p,q}$.

We summarize our current data as follows:

- $U = \operatorname{Spec}(A)$ is a smooth, affine variety with generic point η.
- $Z \cap U$ is a smooth, affine, closed subvariety of U, which is of finite type and of codimension p with generic point z. We have $Z \cap U \cong \operatorname{Spec}(A/(f_1, \dots, f_p))$, where $f_1, \dots, f_p \in A$ is a regular sequence. In particular, as $Z \cap U$ is integral, we can say that $(f_1, \dots, f_p) \subseteq A$ is a prime ideal.
- The non-vanishing loci $U_i := D(f_i)$ form an open covering $\mathfrak{U}$ of $U \setminus Z$, which again is an open subset of U.

Recall from the description in Construction 2.4.1 that the element $\{f_1, \dots, f_p\} \in K_p^M(\mathscr{O}_X(U_{\{1,\dots,p\}}))$ induces a Čech cocycle $\mu \in C^{p-1}(\mathfrak{U}, \mathscr{K}_U^p)$ and we need to follow μ through the composition

$$\check{\mathrm{H}}^{p-1}(\mathfrak{U}, \mathscr{K}_U^p) \to \mathrm{H}^{p-1}(U \setminus Z, \mathscr{K}_U^p) \to \mathrm{H}^p_{Z \cap U}(U, \mathscr{K}_U^p).$$

So first follow the morphism $\check{\mathrm{H}}^{p-1}(\mathfrak{U}, \mathscr{K}_U^p) \to \mathrm{H}^{p-1}(U \setminus Z, \mathscr{K}_U^p)$ from Čech cohomology to sheaf cohomology as described via the naive double complex (B.2), where we use the flasque resolution $\mathscr{K}_U^p \to \mathscr{K}_U^{p,\bullet}$. As the notation is pretty heavy, we visualize the first steps:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r, "\delta"] & C^{p-3}(\mathfrak{U}, \mathscr{K}_U^p) \arrow[r, "\delta"] \arrow[d, "d"] & C^{p-2}(\mathfrak{U}, \mathscr{K}_U^p) \arrow[r, "\delta"] \arrow[d, "d"] & C^{p-1}(\mathfrak{U}, \mathscr{K}_U^p) \arrow[r, "\delta"] \arrow[d, "d"] \arrow[d, dashed, bend right, "\mu \mapsto \theta_0"'] & \cdots \\
\cdots \arrow[r, "\delta"] & C^{p-3}(\mathfrak{U}, \mathscr{K}_U^{p,0}) \arrow[r, "\delta"] \arrow[d, "d"] & C^{p-2}(\mathfrak{U}, \mathscr{K}_U^{p,0}) \arrow[r, "\delta"] \arrow[d, "d"] & C^{p-1}(\mathfrak{U}, \mathscr{K}_U^{p,0}) \arrow[r, "\delta"] \arrow[d, "d"] \arrow[dl, dashed, "\theta_0 \mapsto \theta_1"'] & \cdots \\
\cdots \arrow[r, "\delta"] & C^{p-3}(\mathfrak{U}, \mathscr{K}_U^{p,1}) \arrow[r, "\delta"] \arrow[d, "d"] & C^{p-2}(\mathfrak{U}, \mathscr{K}_U^{p,1}) \arrow[r, "\delta"] \arrow[d, "d"] \arrow[dl, dashed, "\theta_1 \mapsto \theta_2"'] & C^{p-1}(\mathfrak{U}, \mathscr{K}_U^{p,1}) \arrow[r, "\delta"] \arrow[d, "d"] & \cdots \\
\cdots \arrow[r, "\delta"] & C^{p-3}(\mathfrak{U}, \mathscr{K}_U^{p,2}) \arrow[r, "\delta"] \arrow[d, "d"] \arrow[dl, dashed, "\theta_2 \mapsto \theta_3"'] & C^{p-2}(\mathfrak{U}, \mathscr{K}_U^{p,2}) \arrow[r, "\delta"] \arrow[d, "d"] & C^{p-1}(\mathfrak{U}, \mathscr{K}_U^{p,2}) \arrow[r, "\delta"] \arrow[d, "d"] & \cdots \\
\cdots & \vdots & \vdots & \vdots &
\end{tikzcd}$$

Here we inductively construct cocycles in the naive double complex $C^\bullet(\mathfrak{U}, \mathscr{K}_U^{p,\bullet})$

$$\theta_i := \{\theta_{i,I} \in \mathscr{K}_U^{p,i}(U_I) \mid |I| = p - i\} \in C^{p-i-1}(\mathfrak{U}, \mathscr{K}_U^{p,i}).$$

for $i = 0, \ldots, p-1$.

The element for $i = 0$ is induced by $\{f_1, \ldots, f_p\} \in K_p^M(\mathcal{O}_X(U_{\{1,\ldots,p\}}))$ and is clearly given by

$$\theta_0 = \{\theta_{0,\{1,\ldots,p\}} = \{f_{1,\eta}, \ldots, f_{p,\eta}\} \in \mathscr{K}_U^{p,0}(U_{\{1,\ldots,p\}}) = K_p^M(k(\eta)) \otimes \mathbb{Q}\} \in C^{p-1}(\mathfrak{U}, \mathscr{K}_U^{p,0}),$$

where η is the generic point of X (and hence of U).

We now choose a lift $\vartheta_1 \in C^{p-2}(\mathfrak{U}, \mathscr{K}_U^{p,0})$ under δ by setting

$$\vartheta_{1,J} = \begin{cases} \{f_{1,\eta}, \ldots, f_{p,\eta}\} \in \mathscr{K}_U^{p,0}(U_J) = K_p^M(k(\eta)) \otimes \mathbb{Q}, & \text{for } J = \{2, \ldots, p\} \\ 0, & \text{for } J \neq \{2, \ldots, p\}. \end{cases}$$

We clearly have $\delta(\vartheta_1) = \theta_0$, and we need to compute $\theta_1 = d(\vartheta_1)$, i.e. it suffices to look at $d(\vartheta_{1,\{2,\ldots,p\}})$ under the map

$$d \colon \mathscr{K}_U^{p,0}(U_{\{2,\ldots,p\}}) = K_p^M(k(\eta)) \otimes \mathbb{Q} \to \bigoplus_{x \in U^{(1)}_{\{2,\ldots,p\}}} K_{p-1}^M(k(x)) \otimes \mathbb{Q} \cong \mathscr{K}_U^{p,1}(U_{\{2,\ldots,p\}}).$$

Let $x \in U^{(1)}_{\{2,\ldots,p\}}$. By Lemma 2.4.3, we have $d(\{f_{1,\eta}, \ldots, f_{p,\eta}\}) = 0$ if $x \in U_1 = D(f_1)$. On the other hand if $x \in V(f_1)$, then x is already a generic point of $V(f_1) \cap U_{\{2,\ldots,p\}}$, as $V(f_1)$ is of pure codimension 1 by Lemma 2.4.4. Denote by Γ^1 the set of generic points of $V(f_1) \cap U_{\{2,\ldots,p\}}$. Then by Lemma 2.4.3 we obtain

$$\theta_{1,I} = \begin{cases} \mu^1 \cdot \{f_2^{(1)}, \ldots, f_p^{(1)}\} \in \mathscr{K}_U^{p,1}(U_I), & \text{for } I = \{2, \ldots, p\} \\ 0, & \text{for } I \neq \{2, \ldots, p\} \end{cases},$$

where $\mu^1 \cdot \{f_2^{(1)}, \ldots, f_p^{(1)}\} \in \bigoplus_{x \in U^{(1)}_{\{2,\ldots,p\}}} K_{p-1}^M(k(x)) \otimes \mathbb{Q}$ at a summand x is given by

$$(\mu^1 \cdot \{f_2^{(1)}, \ldots, f_p^{(1)}\})_x = \begin{cases} \operatorname{ord}_{\overline{\{x\}}}(f_{1,\eta}) \cdot \{f_2^{(1),x}, \ldots, f_p^{(1),x}\}, & \text{if } x \in \Gamma^1 \\ 0, & \text{else.} \end{cases}$$

Here $f_i^{(1),x}$ denotes the residue of $f_{i,\eta}$ in $k(x)$.

We continue in this fashion inductively. For clarification we want to do another step explicitly. We now need to lift θ_1 to

$$\vartheta_2 \in C^{p-3}(\mathfrak{U}, \mathscr{K}_U^{p,1}) = \prod_{i_0<\cdots<i_{p-3}} \bigoplus_{x \in U^{(1)}_{\{i_0,\ldots,i_{p-3}\}}} K^M_{p-1}(k(x)) \otimes \mathbb{Q}.$$

Again we do this by defining

$$\vartheta_{2,J} = \begin{cases} \vartheta_{2,\{3,\ldots,p\}}, & \text{for } J = \{3,\ldots,p\} \\ 0, & \text{for } J \neq \{3,\ldots,p\}, \end{cases}$$

where $(\vartheta_{2,\{3,\ldots,p\}})_x = (\mu^1 \cdot \{f_2^{(1)}, \ldots, f_p^{(1)}\})_x$ for any $x \in U_{\{2,\ldots,p\}}$, and 0 else. We obtain $\theta_2 \in C^{p-3}(\mathfrak{U}, \mathscr{K}_U^{p,2})$ with $\theta_{2,I} = 0$ for $I \neq \{3,\ldots,p\}$ and $\theta_{2,\{3,\ldots,p\}} = d(\vartheta_{2,\{3,\ldots,p\}})$. Again by Lemma 2.4.3 and 2.4.4 we have that $\theta_{2,\{3,\ldots,p\}} \in \mathscr{K}_U^{p,2}(U_{\{3,\ldots,p\}})$ is zero at any point outside of Γ^2, where Γ^2 is the set of generic points of $V(f_1, f_2) \cap U_{\{3,\ldots,p\}}$. For any point $y \in \Gamma^2$, we obtain by applying Lemma 2.4.3 again, that

$$(\theta_{2,\{3,\ldots,p\}})_y = \sum_{\substack{x \in \Gamma^1, \\ \text{s.t. } y \in \overline{\{x\}}}} \mathrm{ord}_{\overline{\{x\}}}(f_{1,\eta}) \cdot \mathrm{ord}_{\overline{\{y\}}}(f_2^{(1),x}) \cdot \{f_3^{(2),x,y}, \ldots, f_p^{(2),x,y}\},$$

where $f_i^{(2),x,y}$ denotes the residue class of $f_i^{(1),x}$ taken in $k(y) = \mathcal{O}_{\overline{\{x\}},y}/\mathfrak{m}_{\overline{\{x\}},y}$.

We proceed like this inductively to finally obtain an element

$$\theta_{p-1} = \{\theta_{p-1,\{1\}}, \ldots, \theta_{p-1,\{p\}}\} \in C^0(\mathfrak{U}, \mathscr{K}_U^{p,p-1})$$

with $\theta_{p-1,\{i\}} = 0$ for $i \leq p-1$. By Remark B.2 i.) this can be glued to give an element $\theta \in \mathscr{K}_U^{p,p-1}(U \setminus Z)$, which induces our wanted $[\theta] \in \mathrm{H}^{p-1}(U \setminus Z, \mathscr{K}^p_{U \setminus Z})$.

To finally obtain $c(Z)_\alpha \in \mathrm{H}^p_{Z \cap U}(U, \mathscr{K}_U^p)$, we need to understand the connecting morphism $\mathrm{H}^{p-1}(U \setminus Z, \mathscr{K}^p_{U \setminus Z}) \to \mathrm{H}^p_{Z \cap U}(U, \mathscr{K}_U^p)$ in the long exact sequence of cohomology with supports as in Corollary A.6. For this consider the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Gamma_{Z \cap U}(U, \mathscr{K}_U^{p,p-1}) & \longrightarrow & \Gamma(U, \mathscr{K}_U^{p,p-1}) & \longrightarrow & \Gamma(U \setminus Z, \mathscr{K}_{U \setminus Z}^{p,p-1}) & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
0 & \longrightarrow & \Gamma_{Z \cap U}(U, \mathscr{K}_U^{p,p}) & \longrightarrow & \Gamma(U, \mathscr{K}_U^{p,p}) & \longrightarrow & \Gamma(U \setminus Z, \mathscr{K}_{U \setminus Z}^{p,p}) & \longrightarrow & 0,
\end{array}$$

which resolves explicitly to

$$\begin{tikzcd}
0 \arrow[r] & \bigoplus\limits_{x\in U^{(p-1)}\cap Z} k(x)^*\otimes\mathbb{Q} \arrow[r] \arrow[d] & \bigoplus\limits_{x\in U^{(p-1)}} k(x)^*\otimes\mathbb{Q} \arrow[r, "\cong"] \arrow[d, "d_U"] & \bigoplus\limits_{x\in (U\setminus Z)^{(p-1)}} k(x)^*\otimes\mathbb{Q} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \bigoplus\limits_{x\in U^{(p)}\cap Z} \mathbb{Z}\otimes\mathbb{Q} \arrow[r, hook, "\beta"] & \bigoplus\limits_{x\in U^{(p)}} \mathbb{Z}\otimes\mathbb{Q} \arrow[r] & \bigoplus\limits_{x\in (U\setminus Z)^{(p)}} \mathbb{Z}\otimes\mathbb{Q} \arrow[r] & 0.
\end{tikzcd}$$

The upper right horizontal morphism is an isomorphism by Lemma 2.3.4 (in particular $\bigoplus\limits_{x\in U^{(p-1)}\cap Z} k(x)^*\otimes\mathbb{Q} = 0$ in the upper left corner). Then a representative of $c(Z)_\alpha$ in $\bigoplus_{x\in U^{(p)}\cap Z}\mathbb{Z}\otimes\mathbb{Q}\cong\mathbb{Z}\otimes\mathbb{Q}$ (see Lemma 2.3.8) is given by taking a preimage of $d_U(\theta)$ under the map β. Hence the Proposition follows if we can show that

$$(d_U(\theta))_x = \begin{cases} 0, & \text{for } x\neq z \\ 1, & \text{for } x = z, \end{cases} \tag{2.4.2}$$

where z is the generic point of $Z\cap U$. By codimension reasons for $x\in U^{(p)}\setminus\{z\}$, we must have $x\notin Z = V(f_1,\dots,f_p)$, hence it is clear that $(d_U(\theta))_x = 0$. Now let $x = z$. By Lemma 2.4.4, the generic point z lies in exactly one irreducible component $Y_i = \overline{\{y_i\}}$ of $V(f_1,\dots,f_i)$ for $1\leq i\leq p-1$, with $y_i\in U_{\{i+1,\dots,p\}}$. Thus by construction of θ, we have

$$(d_U(\theta))_z = \mathrm{ord}_{Y_1}(f_{1,\eta})\cdot\mathrm{ord}_{Y_2}(f_2^{(1),y_1})\cdots\mathrm{ord}_{Y_{p-1}}(f_{p-1}^{(p-2),y_1,\dots,y_{p-2}})\cdot\mathrm{ord}_Z(f_p^{(p-1),y_1,\dots,y_{p-1}}),$$

where the f_i's successively pass through the residue fields. Hence by Remark 2.4.7 each of the factors in the product equals 1. □

Remark 2.4.8. By (2.3.5) for any open $U\subseteq X$ we have

$$\mathrm{H}^p_{Z\cap U}(U,\mathscr{K}^p_X|_U) = \Gamma_{Z\cap U}(U,\mathscr{K}^{p,p}_X|_U) = \mathcal{H}^0_Z(\mathscr{K}^{p,p}_X)(U),$$

where the right hand side is the subsheaf with supports in Z as in Lemma A.2 i.). Hence as the restriction maps of the presheaf $U\mapsto \mathrm{H}^p_{Z\cap U}(U,\mathscr{K}^p_X|_U)$ behave equivalently as in $\mathcal{H}^0_Z(\mathscr{K}^{p,p}_X)$, this presheaf is already a sheaf on X. Thus by the local description (2.4.2) in the above Proposition 2.4.2, we can glue the family $\{c(Z)_\alpha\in\mathrm{H}^p_{Z\cap U_\alpha}(U_\alpha,\mathscr{K}^p_{U_\alpha})\}_\alpha$ to an element $c(Z)\in\mathrm{H}^p_Z(X,\mathscr{K}^p_X)$. Together with Remark 2.3.9 and Proposition 2.4.2, we immediately obtain the following important result.

Corollary 2.4.9. Let the notation be as above. The image of the class $c(Z)$ under the morphism

$$\mathrm{H}^p_Z(X, \mathscr{K}^p_X) \to \mathrm{H}^p(X, \mathscr{K}^p_X)$$

coincides with $\mathrm{cl}_{\mathrm{univ}}([Z])$.

3

Differential Forms on the Berkovich Analytification

In this chapter we develop the notion of differential forms on the Berkovich analytification X^{an} of an algebraic variety X over an algebraically closed field K which is complete with respect to a nontrivial, non-Archimedean absolute value. We begin by introducing superforms in the sense of Lagerberg [Lag12] on $\mathbb{R}^r$, and obtain differential forms on X^{an} by locally pulling them back along tropical charts. We follow the approach of Gubler via algebraic moment maps as in [Gub16].

3.1 Superforms on $\mathbb{R}^r$ and on polyhedra

Remark 3.1.1. Let N be a free abelian group of rank r with dual abelian group $M := \mathrm{Hom}(N, \mathbb{Z})$ and associated real vector spaces $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$ respectively $M_{\mathbb{R}}$ of dimension r. The choice of a $\mathbb{Z}$-basis of N induces an isomorphism $N \cong \mathbb{Z}^r, N_{\mathbb{R}} \cong \mathbb{R}^r, M_{\mathbb{R}} \cong \mathbb{R}^{r*} := \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^r, \mathbb{R})$. For our following constructions we will focus on the case $N = \mathbb{Z}^r$ with standard basis $e_1, \ldots, e_r$. For a coordinate-free approach, the algebraic torus $\mathrm{Spec}(K[N])$ with character group N takes the role of the torus $\mathrm{Spec}(K[T_1^{\pm 1}, \ldots, T_r^{\pm 1}])$ of rank r, as it is done in [Gub16].

Definition 3.1.2.

i.) For an open subset $U \subseteq \mathbb{R}^r$ we denote by $A^p(U)$ the space of smooth real differential forms of degree p. We define the space of *superforms of bidegree* (p, q) on U as

$$A^{p,q}(U) := A^p(U) \otimes_{\mathcal{C}^\infty(U)} A^q(U) = A^p(U) \otimes_{\mathbb{R}} \Lambda^q \mathbb{R}^{r*} = \mathcal{C}^\infty(U) \otimes_{\mathbb{R}} \Lambda^p \mathbb{R}^{r*} \otimes_{\mathbb{R}} \Lambda^q \mathbb{R}^{r*}.$$

ii.) With choice of a basis $x_1, \dots, x_r$ of $\mathbb{R}^r$ we can formally write a superform $\alpha \in A^{p,q}(U)$ as

$$\alpha = \sum_{|I|=p,|J|=q} \alpha_{IJ} d'x_I \wedge d''x_J,$$

where $I = \{i_1, \dots, i_p\}$ respectively $J = \{j_1, \dots, j_q\}$ are ordered subsets of $\{1, \dots, r\}$, $\alpha_{IJ} \in \mathcal{C}^\infty(U)$ are smooth functions and

$$d'x_I \wedge d''x_J := (dx_{i_1} \wedge \dots \wedge dx_{i_p}) \otimes_{\mathbb{R}} (dx_{j_1} \wedge \dots \wedge dx_{j_q}).$$

iii.) We define the wedge product

$$A^{p,q}(U) \times A^{p',q'}(U) \to A^{p+p',q+q'}(U)$$
$$(\alpha, \beta) \mapsto \alpha \wedge \beta$$

in coordinates as

$$\begin{aligned} \alpha \wedge \beta &:= \left(\sum_{|I|=p,|J|=q} \alpha_{IJ} d'x_I \wedge d''x_J \right) \wedge \left(\sum_{|K|=p',|L|=q'} \beta_{KL} d'x_K \wedge d''x_L \right) \\ &:= (-1)^{p'q} \sum_{|I|=p,|J|=q,|K|=p',|L|=q'} \alpha_{IJ}\beta_{KL} (d'x_I \wedge d'x_K) \wedge (d''x_J \wedge d''x_L), \end{aligned}$$

where $d'x_I \wedge d'x_K \in \Lambda^{p+p'}\mathbb{R}^{r*}$ respectively $d''x_J \wedge d''x_L \in \Lambda^{q+q'}\mathbb{R}^{r*}$ is the usual wedge product.

iv.) There is a *differential operator*

$$d' \colon A^{p,q}(U) = A^p(U) \otimes_{\mathbb{R}} \Lambda^q \mathbb{R}^{r*} \to A^{p+1}(U) \otimes_{\mathbb{R}} \Lambda^q \mathbb{R}^{r*} = A^{p+1,q}(U)$$

given by $D \otimes_{\mathbb{R}} \mathrm{id}$ where D is the usual exterior derivative on $A^p(U)$. Also note that $A^{p,q}(U) = \Lambda^p \mathbb{R}^{r*} \otimes_{\mathbb{R}} A^q(U)$, and we define a second operator $d'' := (-1)^p \mathrm{id} \otimes_{\mathbb{R}} D$. In coordinates this gives

$$d' \left(\sum_{|I|=p,|J|=q} \alpha_{IJ} d'x_I \wedge d''x_J \right) = \sum_{|I|=p,|J|=q} \sum_{i=1}^r \frac{\partial \alpha_{IJ}}{\partial x_i} d'x_i \wedge d'x_I \wedge d''x_J$$

and

$$d'' \left(\sum_{|I|=p,|J|=q} \alpha_{IJ} d'x_I \wedge d''x_J \right) = (-1)^p \sum_{|I|=p,|J|=q} \sum_{i=1}^r \frac{\partial \alpha_{IJ}}{\partial x_i} d'x_I \wedge d''x_i \wedge d''x_J.$$

Finally we define $d := d' + d''$.

Remark 3.1.3. As in differential geometry, we may view a superform

$$\alpha = \sum_{i=1}^{n} \alpha_i \otimes \omega_i \otimes \mu_i \in A^{p,q}(U) = \mathcal{C}^\infty(U) \otimes \Lambda^p \mathbb{R}^{r*} \otimes \Lambda^q \mathbb{R}^{r*}$$

at a point $x \in U$ as a multilinear map

$$(\mathbb{R}^r)^{p+q} \to \mathbb{R},\ (n_1, \ldots, n_{p+q}) \mapsto \sum_{i=1}^{n} \alpha_i(x)\omega_i(n_1, \ldots, n_p)\mu_i(n_{p+1}, \ldots, n_{p+q})$$

which is alternating in $(n_1, \ldots, n_p)$ and $(n_{p+1}, \ldots, n_{p+q})$. We write $\langle \alpha(x); n_1, \ldots, n_{p+q}\rangle$ for a superform α and such an evaluation at $x \in U$ and $n_1, \ldots, n_{p+q} \in \mathbb{R}^r$.

Remark 3.1.4.

i.) For superforms α of degree (p, q) and β of degree (p', q') one computes easily the relations

$$d'(\alpha \wedge \beta) = d'\alpha \wedge \beta + (-1)^{p+q}\alpha \wedge d'\beta$$

and similarly

$$d''(\alpha \wedge \beta) = d''\alpha \wedge \beta + (-1)^{p+q}\alpha \wedge d''\beta.$$

Hence the choice of sign in d''.

ii.) Note that we have as usual $d'(d'\alpha) = 0$ and $d''(d''\alpha) = 0$, however in general not $d'(d''\alpha) = 0$. Indeed, for $\mathbb{R}^2$ with coordinates x, y consider the superform $xy \in A^{0,0}(\mathbb{R}^2) = \mathcal{C}^\infty(\mathbb{R}^2)$. Then $d'(d''(xy)) = d'(yd''x + xd''y) = d'y \wedge d''x + d'x \wedge d''y \neq 0$.

Remark 3.1.5.

i.) Let $F\colon \mathbb{R}^{r'} \to \mathbb{R}^r$ with $F(x) = f(x) + a$ be an affine map. Here f is the corresponding linear map and $a \in \mathbb{R}^r$. Furthermore let $U' \subseteq \mathbb{R}^{r'}$ and $U \subseteq \mathbb{R}^r$ with $F(U') \subseteq U$. Note that f induces a linear map $f^*\colon \mathbb{R}^{r*} \to \mathbb{R}^{r'*}$ which again induces a linear map $f^*\colon \Lambda^k\mathbb{R}^{r*} \to \Lambda^k\mathbb{R}^{r'*}$. In particular we obtain a well-defined *pullback morphism*

$$F^*\colon A^{p,q}(U) = \mathcal{C}^\infty(U) \otimes_{\mathbb{R}} \Lambda^p\mathbb{R}^{r*} \otimes_{\mathbb{R}} \Lambda^q\mathbb{R}^{r*} \to A^{p,q}(U')$$
$$g \otimes \omega \otimes \mu \mapsto (g \circ F) \otimes f^*\omega \otimes f^*\mu.$$

ii.) Note that with the representation $A^{p,q}(U) = A^p(U) \otimes_{\mathbb{R}} \Lambda^q \mathbb{R}^{r*}$ the pullback for an affine map F as above can be written as

$$F^*\colon A^p(U) \otimes_{\mathbb{R}} \Lambda^q \mathbb{R}^{r*} \to A^p(U') \otimes_{\mathbb{R}} \Lambda^q \mathbb{R}^{r'*}$$
$$\omega \otimes_{\mathbb{R}} \mu \mapsto F^*\omega \otimes_{\mathbb{R}} f^*\mu,$$

where $F^*\omega$ is the usual pullback of smooth p-forms with respect to the smooth function F. In particular we obtain the corresponding result that F^* commutes with d', d'' and d.

iii.) For $n'_1, \dots, n'_{p+q} \in \mathbb{R}^{r'}$ and $x' \in U'$ the evaluation as in Remark 3.1.3 of the pullback for $\alpha \in A^{p,q}(U)$ can be written as

$$\langle F^*\alpha(x'); n'_1, \dots, n'_{p+q}\rangle = \langle \alpha(F(x')); f(n'_1), \dots, f(n'_{p+q})\rangle.$$

iv.) Let $F\colon U' \to U$ be a smooth map where $U \subseteq \mathbb{R}^r$ and $U' \subseteq \mathbb{R}^{r'}$ are open subsets. We can define a 'naive' pullback

$$F^*\colon A^{p,q}(U) = A^p(U) \otimes_{\mathcal{C}^\infty(U)} A^q(U) \to A^p(U') \otimes_{\mathcal{C}^\infty(U')} A^q(U') = A^{p,q}(U'),$$

which is just given by the tensor product of the usual pullbacks of differential p- respectively q-forms. This construction and the definition in i.) match for affine maps, however in general for smooth maps it does not commute with d', d'', d. Indeed, let $U' = \mathbb{R}^2$ and $U = \mathbb{R}$ and $F(x,y) = xy$ and t the coordinate of $\mathbb{R}$, then $d'F^*(d''t) = d'(yd''x + xd''y) = d'y \wedge d''x + d'x \wedge d''y \neq 0$, but $d'(d''t) = 0$ and hence $d'F^*(d''t) \neq F^*(d'(d''t))$.
The reason is that $d' = D \otimes \mathrm{id}$, but the pullback on the second factor uses the differential of F at the point $x \in \mathbb{R}^{r'}$, which might depend on x. In the affine case however, the differential has no such dependence.

Remark 3.1.6. Note that for all p, q the functor $U \mapsto A^{p,q}(U)$ is a sheaf on $\mathbb{R}^r$ with $A^{p,q} = 0$ for $\max(p,q) > r$. Furthermore we have $A^{0,0}(U) = C^\infty(U)$.

Definition 3.1.7. Let U be an open subset of $\mathbb{R}^r$ and $\alpha \in A^{p,q}(U)$. The *support* of α is defined in the sheaf-theoretic sense, i.e.

$$\mathrm{supp}(\alpha) = U \setminus \{x \in U \mid \exists \text{ open } U_x \text{ around } x \text{ with } \alpha|_{U_x} = 0\}.$$

A superform has *compact support* if its support is a compact set. Denote by $A_c^{p,q}(U)$ the space of (p,q)-superforms with compact support.

Definition 3.1.8. Let U be an open subset of $\mathbb{R}^r$ and $\alpha \in A_c^{r,r}(U)$. Choose a basis $x_1, \ldots, x_r$ of the $\mathbb{Z}$-module $\mathbb{Z}^r$. Then α can be written in the form

$$\alpha = f_\alpha d'x_1 \wedge \cdots \wedge d'x_r \wedge d''x_1 \wedge \cdots \wedge d''x_r$$

with $f_\alpha \in C_c^\infty(U)$. We then define

$$\int_U \alpha := (-1)^{\frac{r(r-1)}{2}} \int_U f_\alpha,$$

where the right hand side is the integral with respect to the volume of the lattice $\mathbb{Z}^r \subseteq \mathbb{R}^r$.

Remark 3.1.9. In Definition 3.1.8 the function f_α does not rely on the choice of the *integral* basis $x_1, \ldots, x_r$. Indeed, keeping the above notation, let $\alpha \in A_c^{r,r}(U)$ and $y_1, \ldots, y_r$ another $\mathbb{Z}$-basis of $\mathbb{Z}^r$. Furthermore let $g_\alpha \in C_c^\infty(U)$ with

$$\alpha = g_\alpha d'y_1 \wedge \cdots \wedge d'y_r \wedge d''y_1 \wedge \cdots \wedge d''y_r$$

The change of basis matrix $B \in M_r(\mathbb{Z})$ satisfies $\det(B) \in \{\pm 1\}$ and by standard differential geometry we obtain

$$\begin{aligned} g_\alpha d'y_1 \wedge \cdots \wedge d'y_r \wedge d''y_1 \wedge \cdots \wedge d''y_r &= \\ g_\alpha(\det(B) d'x_1 \wedge \cdots \wedge d'x_r) \wedge (\det(B) d''x_1 \wedge \cdots \wedge d''x_r) &= \\ \det(B)^2 g_\alpha d'x_1 \wedge \cdots \wedge d'x_r \wedge d''x_1 \wedge \cdots \wedge d''x_r &= \\ g_\alpha d'x_1 \wedge \cdots \wedge d'x_r \wedge d''x_1 \wedge \cdots \wedge d''x_r&. \end{aligned}$$

In particular we get $g_\alpha = f_\alpha$.

Proposition 3.1.10. (Change of Variables Formula) Let $F\colon \mathbb{R}^r \to \mathbb{R}^r$ be an affine map with linear part represented by $A \in M_r(\mathbb{R})$. Let U be an open subset of $\mathbb{R}^r$ and $\alpha \in A_c^{r,r}(U)$. Then

$$\int_{F^{-1}(U)} F^*\alpha = |\det(A)| \int_U \alpha.$$

Proof. The Jacobi matrix of F is precisely A. Note that by Remark 3.1.5 we have

$$\langle F^*\alpha(x); n_1, \ldots, n_{r+r} \rangle = \langle \alpha(F(x)); An_1, \ldots, An_{r+r}) \rangle = \det(A)^2 \cdot \langle \alpha(F(x)); n_1, \ldots, n_{r+r} \rangle,$$

as $\eta(An_1, \ldots An_r) = \det(A) \cdot \eta(n_1, \ldots, n_r)$ for any alternating r-Form η on $\mathbb{R}^r$. Now if

$\det(A) = 0$, there is nothing to show. Otherwise F is an isomorphism, in particular diffeomorphism, and the usual change of variables formula yields

$$\int_{F^{-1}(U)} F^*\alpha = (-1)^{\frac{r(r-1)}{2}} \int_{F^{-1}(U)} \det(A)^2 (f_\alpha \circ F) =$$
$$|\det(A)|(-1)^{\frac{r(r-1)}{2}} \int_{F^{-1}(U)} |\det(A)|(f_\alpha \circ F) = |\det(A)| \int_U \alpha.$$

□

Remark 3.1.11. It follows from the change of variables formula above that the definition of the integral depends on the underlying integral $\mathbb{R}$-affine structure of $\mathbb{R}^n$, but not on a chosen orientation.

Remark 3.1.12. Integration can also be described in terms of a contraction: Let $\alpha \in A^{p,q}(U)$ as usual and let $I \subseteq \{1, \ldots, p+q\}$ be a subset of cardinality $|I| = s$ with s' elements contained in $\{1, \ldots, p\}$ and thus $s'' = s - s'$ elements contained in $\{p+1, \ldots, p+q\}$. Given vectors $v_1, \ldots, v_s \in \mathbb{R}^r$, the *contraction*

$$\langle \alpha; v_1, \ldots, v_s \rangle_I \in A^{p-s', q-s''}(U)$$

is given by inserting $v_1, \ldots, v_s$ for the variables $(n_i)_{i \in I}$, where we view α as a multilinear map as in Remark 3.1.3.

Now if $\alpha \in A_c^{r,r}(U)$ and using the standard basis $e_1, \ldots, e_r$ of $\mathbb{Z}^r$, the contraction

$$\langle \alpha; e_1, \ldots, e_r \rangle_{\{r+1, \ldots, 2r\}}$$

is an $(r, 0)$-superform which we can view as a classical r-form on U. Clearly then

$$\int_U \alpha = (-1)^{\frac{r(r-1)}{2}} \int_U \langle \alpha; e_1, \ldots, e_r \rangle_{\{r+1, \ldots, 2r\}}$$

where on the right hand side the integral of usual r-forms is taken. The contraction can of course also be performed on the first r variables and then do as above for a contraction in $A_c^{0,r}(U)$.

Next we are going to describe integration of superforms along polyhedra and their boundaries. Before this though, we quickly recall needed definitions in basic convex geometry.

Definition 3.1.13. (Review of basic convex geometry)

i.) A *polyhedron* $\sigma \subseteq \mathbb{R}^r$ is the intersection of finitely many halfspaces $H_i = \{w \in \mathbb{R}^r \mid \langle u_i, w \rangle \leq c_i\}$ with $c_i \in \mathbb{R}$ and $u_i \in \mathbb{R}^{r*}$, $i \in \{1, \ldots, n\}$. A *polytope* is a bounded polyhedron.

ii.) We say that σ is an *integral* Γ*-affine polyhedron* for an additive subgroup Γ of $\mathbb{R}$ if we may choose all $u_i \in \mathbb{Z}^{r*}$ and $c_i \in \Gamma$.

iii.) Let $J = \{j \in \{1, \dots, n\} \mid \langle u_j, w\rangle = c_j \ \forall w \in \sigma\}$. Then

$$\mathbb{A}_\sigma = \{x \in \mathbb{R}^r \mid \langle u_j, x\rangle = c_j \ \forall j \in J\}$$

is the smallest affine subspace of $\mathbb{R}^r$ which contains σ. Its underlying linear subspace is $\mathbb{L}_\sigma = \{x \in \mathbb{R}^r \mid \langle u_j, x\rangle = 0 \ \forall j \in J\}$. The *dimension of* σ is $\dim \sigma := \dim_{\mathbb{R}} \mathbb{L}_\sigma$.

iv.) In particular for an integral Γ-affine polyhedron σ (recall that $\ker(A) \cap \mathbb{Z}^r$ is a full-rank lattice in $\ker(A)$ for a matrix A with integral coefficients) we obtain a lattice $\mathbb{Z}_\sigma := \mathbb{L}_\sigma \cap \mathbb{Z}^r$ in $\mathbb{L}_\sigma$.

v.) A *closed face* ρ of a polyhedron σ (denoted by $\rho \prec \sigma$) is either σ itself or an intersection $\sigma \cap \partial H$, where ∂H is the boundary of some halfspace H containing σ. In particular a closed face is a polyhedron again. Also note that the empty set is a closed face as well. An *open face* of a polyhedron is a closed face without all its properly contained closed faces.

Definition 3.1.14. Let $\sigma \subseteq \mathbb{R}^r$ be a polyhedron of dimension n, and choose a $\mathbb{Z}$-basis of the lattice $\mathbb{Z}_\sigma$. This is also an $\mathbb{R}$-basis of $\mathbb{L}_\sigma$. In particular this basis induces an $\mathbb{R}$-affine map $G \colon \mathbb{R}^n \to \mathbb{A}_\sigma$. We get an integral $\int_\sigma \alpha$ for any $\alpha \in A_c^{n,n}(U)$, where U is an open neighborhood of σ, by setting

$$\int_\sigma \alpha := \int_{G^{-1}(U)} G^*\alpha,$$

where on the right hand side we integrate the pullback $G^*\alpha \in A_c^{n,n}(G^{-1}(U))$ along $G^{-1}(U) \subseteq \mathbb{R}^n$ as in Definition 3.1.8. By Proposition 3.1.10, this integral is well-defined.

Remark 3.1.15. Let $H = \{w \in \mathbb{R}^r \mid \langle u, w\rangle \leq c\}$ be an integral $\mathbb{R}$-affine halfspace in $\mathbb{R}^r$ with nontrivial $u \in \mathbb{Z}^{r*}$ and $c \in \mathbb{R}$. Using a translation, we may assume that $c = 0$ and the boundary ∂H is an $r-1$-dimensional linear subspace of $\mathbb{R}^r$. Let $[\omega_{\partial H,H}]$ be the generator of the $\mathbb{Z}$-module $\mathbb{Z}^r/(\mathbb{Z}^r \cap \partial H) \cong \mathbb{Z}$ which 'points outwards', i.e. there clearly is a $u_{\partial H,H} \in \mathbb{Z}^{r*}$ such that $\langle u_{\partial H,H}, w\rangle \leq 0$ for all $w \in H$ and $\langle u_{\partial H,H}, \omega_{\partial H,H}\rangle = 1$. Here we choose a representative $\omega_{\partial H,H} \in \mathbb{Z}^r$. Also note that $u_{\partial H,H}$ is uniquely determined by the above properties.

Definition 3.1.16. Let $U \subseteq \mathbb{R}^r$ be open and let $\sigma \subseteq U$ be an r-dimensional integral $\mathbb{R}$-affine polyhedron. For any closed face ρ of dimension $r-1$, there is a halfspace H containing σ with $\rho = \sigma \cap \partial H$. Note that the affine hyperplane ∂H is generated by ρ. Now let $\omega_{\rho,\sigma} := \omega_{\partial H,H}$ as in Remark 3.1.15 above, and note that $\omega_{\rho,\sigma}$ is determined up to addition with elements in $\mathbb{Z}_\rho = \mathbb{Z}^r \cap \mathbb{L}_\rho$.

Now let $\eta \in A_c^{r-1,r}(U)$ and consider the contraction $\langle \eta; \omega_{\rho,\sigma} \rangle_{\{2r-1\}} \in A_c^{r-1,r-1}(U)$ as in Remark 3.1.12. As the dimension of ρ is $r-1$, and as η is alternating in its last r factors, the restriction of this contraction to ρ does not depend on the choice of the representative $\omega_{\rho,\sigma}$. We then define the integral of η along the boundary of σ by

$$\int_{\partial\sigma} \eta := \sum_{\rho} \int_{\rho} \langle \eta; \omega_{\rho,\sigma} \rangle_{\{2r-1\}},$$

where the sum on the right hand side runs over all closed faces ρ of σ of dimension $r-1$, and the integrals along ρ are taken as in Definition 3.1.14. Analogously for $\eta \in A_c^{r,r-1}(U)$ we define

$$\int_{\partial\sigma} \eta := \sum_{\rho} \int_{\rho} \langle \eta, \omega_{\rho;\sigma} \rangle_{\{r\}}.$$

Again, as seen before, the integrals do not depend on the choice of an orientation on $\mathbb{R}^r$.

If σ is an integral $\mathbb{R}$-affine polyhedron of any dimension n and if $\eta \in A_c^{n-1,n}(U)$ for an open subset U of $\mathbb{R}^r$ containing σ, then we define $\int_{\partial\sigma} \eta$ by applying the above to the affine space $\mathbb{A}_\sigma$ and to the pullback of η as in Definition 3.1.14.

Remark 3.1.17. Let σ be an integral $\mathbb{R}$-affine polyhedron of dimension n and $\eta \in A_c^{n-1,n}(U)$ for an open subset $U \subseteq \mathbb{R}^r$ containing σ. We can give a concrete description of $\int_{\partial\sigma} \eta$ in terms of integrals over $(n-1)$-forms as follows. For every closed face ρ of σ consider as usual the lattice $\mathbb{Z}_\rho = \mathbb{L}_\sigma \cap \mathbb{Z}^r$. If ρ is of codimension 1, and $e_1^\rho, \ldots, e_{n-1}^\rho$ is a basis of $\mathbb{Z}_\rho$, then $\omega_{\rho,\sigma}, e_1^\rho, \ldots, e_{n-1}^\rho$ form a basis of $\mathbb{Z}_\sigma$. The contraction $\langle \eta; \omega_{\rho,\sigma}, e_1^\rho, \ldots, e_{n-1}^\rho \rangle_{\{n,\ldots,2n-1\}}$ can be viewed as a classical $(n-1)$-form on U and we get

$$\int_{\partial\sigma} \eta = \sum_{\rho} \int_{\rho} \langle \eta; \omega_{\rho,\sigma} \rangle_{\{2r-1\}} = (-1)^{\frac{n(n-1)}{n}} \sum_{\rho} \int_{\rho} \langle \eta; \omega_{\rho,\sigma}, e_1^\rho, \ldots, e_{n-1}^\rho \rangle_{\{n,\ldots,2n-1\}}.$$

We can now formulate Stokes' formula in the situation of superforms on polyhedra.

Proposition 3.1.18. (Stokes' Formula) Let σ be an n-dimensional integral $\mathbb{R}$-affine polyhedron contained in an open subset $U \subseteq \mathbb{R}^r$. For any $\eta' \in A_c^{n-1,n}(U)$ and any $\eta'' \in A_c^{n,n-1}(U)$, we have

$$\int_{\sigma} d'\eta' = \int_{\partial\sigma} \eta', \quad \int_{\sigma} d''\eta'' = \int_{\partial\sigma} \eta''.$$

Proof. See [Gub16, Proposition 2.9]. □

Definition 3.1.19. A *supercurrent* on an open $U \subseteq \mathbb{R}^r$ is a continuous linear functional on $A_c^{p,q}(U)$, and we denote the space of such supercurrents by $D_{p,q}(U)$. The word 'continuous' in this situation does ask for an explanation. The precise definition is slightly arduous, though very similar to the complex case, hence we refer to [Dem12, §2.A] for details. The general idea is to introduce a topology on $A_c^{p,q}(U)$ and then to consider the topological dual. Suffice it to say in our case the following:

i.) We denote the pairing of $T \in D_{p,q}(U)$ and $\alpha \in A^{p,q}(U)$ by $\langle T, \alpha\rangle$.

ii.) There is a canonical embedding $A^{p,q}(U) \hookrightarrow D_{r-p,r-q}(U)$ which maps $\alpha \in A^{p,q}(U)$ to $[\alpha] \in D_{r-p,r-q}(U)$ given by

$$\langle [\alpha], \beta\rangle = \int_U \alpha \wedge \beta$$

for any $\beta \in A_c^{r-p,r-q}(U)$.

iii.) Let $T \in D_{p,q}(U)$. We define supercurrents $d'T \in D_{p-1,q}(U)$ and $d''T \in D_{p,q-1}(U)$ via the action of d' and d'' on forms, i.e.

$$d'T \colon \alpha \mapsto (-1)^{p+q+1}\langle T, d'\alpha\rangle,$$
$$d''T \colon \alpha \mapsto (-1)^{p+q+1}\langle T, d''\alpha\rangle.$$

The signs are chosen such that d' and d'' on forms and currents correspond, i.e. that we have

$$d'[\alpha] = [d'\alpha] \text{ and } d''[\alpha] = [d''\alpha]$$

for $\alpha \in A^{p,q}(U)$.

iv.) A current T is called *d'-closed*, if $d'T = 0$. Analogously for d''.

3.2 Superforms on polyhedral complexes

Definition 3.2.1.

i.) An (integral Γ-affine) *polyhedral complex* $\mathscr{C}$ in $\mathbb{R}^r$ is a finite set of (integral Γ-affine) polyhedra in $\mathbb{R}^r$ which satisfies the following conditions:

a.) If $\sigma \in \mathscr{C}$, then all closed faces of σ lie in $\mathscr{C}$.

b.) If $\sigma, \tau \in \mathscr{C}$, then $\sigma \cap \tau$ is a closed face of both.

ii.) The *support* $|\mathscr{C}|$ of a polyhedral complex $\mathscr{C}$ is the union of all polyhedra in $\mathscr{C}$.

iii.) A polyhedral complex $\mathscr{C}$ is called *pure dimensional of dimension* n if every maximal polyhedron in $\mathscr{C}$ has dimension n. Write $\mathscr{C}_k := \{\sigma \in \mathscr{C} \mid \dim \sigma = k\}$ for $k \in \mathbb{N}$.

iv.) A polyhedral complex $\mathscr{D}$ *subdivides* the polyhedral complex $\mathscr{C}$ if they have the same support and every $\delta \in \mathscr{D}$ is contained in some $\sigma \in \mathscr{C}$. We then say $\mathscr{D}$ is a *subdivision of* $\mathscr{C}$.

Definition 3.2.2. Let $\mathscr{C}$ be a polyhedral complex in $\mathbb{R}^r$ and Ω an open subset of $|\mathscr{C}|$.

i.) A *superform* $\alpha \in A^{p,q}_{\mathscr{C}}(\Omega)$ *of bidegree* (p,q) on the polyhedral complex $\mathscr{C}$ is given by a superform $\alpha' \in A^{p,q}(V)$ where $V \subseteq \mathbb{R}^r$ is open and $V \cap |\mathscr{C}| = \Omega$.

ii.) Two forms $\alpha' \in A^{p,q}(V)$ and $\alpha'' \in A^{p,q}(W)$ with $V \cap |\mathscr{C}| = W \cap |\mathscr{C}| = \Omega$ define the same superform in $A^{p,q}_{\mathscr{C}}(\Omega)$ if their *restrictions to any polyhedron in* $\mathscr{C}$ *agree.* That is, for all $\sigma \in \mathscr{C}$ we have

$$\langle \alpha'(x); v_1, \ldots, v_p, w_1, \ldots, w_q \rangle = \langle \alpha''(x); v_1, \ldots, v_p, w_1, \ldots, w_q \rangle$$

for all $x \in \sigma \cap \Omega$ and $v_i, w_j \in \mathbb{L}_\sigma$.
In this case we write $\alpha'|_\sigma = \alpha''|_\sigma$. If $\alpha \in A^{p,q}_{\mathscr{C}}(\Omega)$ is given by $\alpha' \in A^{p,q}(V)$, write $\alpha'|_\Omega = \alpha$.

We see easily that $A^{p,q}_{\mathscr{C}}(\Omega)$ only depends on Ω as an open subset of the support $|\mathscr{C}|$, not on the structure of the polyhedral complex $\mathscr{C}$.

iii.) The *support* $\operatorname{supp}(\alpha)$ of some $\alpha \in A^{p,q}_{\mathscr{C}}(\Omega)$ given by $\alpha' \in A^{p,q}(V)$ is defined as $\operatorname{supp}(\alpha') \cap |\mathscr{C}|$. As usual we denote by $A^{p,q}_{\mathscr{C},c}(\Omega)$ the space of superforms with compact support.

Remark 3.2.3.

i.) The definition of $\wedge, d, d', d''$ on superforms on $\mathbb{R}^r$ carries over to superforms on polyhedral complexes.

ii.) Let $F\colon \mathbb{R}^{r'} \to \mathbb{R}^r$, $F(x) = f(x) + a$ be an affine map with $F(|\mathscr{C}'|) \subseteq |\mathscr{C}|$ for polyhedral complexes $\mathscr{C}' \subseteq \mathbb{R}^{r'}$ and $\mathscr{C} \subseteq \mathbb{R}^r$. Then we have $f(\mathbb{L}_{\sigma'}) \subseteq \mathbb{L}_\sigma$ for all $\sigma' \in \mathscr{C}'$ with $F(\sigma') \subseteq \sigma$ for some $\sigma \in \mathscr{C}$, after passing to some subdivision if necessary. Hence the pullback in Remark 3.1.5 carries over to a pullback $F^*\colon A^{p,q}_{\mathscr{C}}(\Omega) \to A^{p,q}_{\mathscr{C}'}(\Omega')$ for open subsets Ω respectively Ω' in the support of the polyhedral complexes.

Definition 3.2.4. Let $\mathscr{C}$ be a polyhedral complex of pure dimension.

i.) A *weight* on $\mathscr{C}$ is a function m which assigns to every maximal polyhedron $\sigma \in \mathscr{C}$ an integer $m_\sigma \in \mathbb{Z}$. We call $\mathscr{C}$ together with m a *weighted polyhedral complex* $(\mathscr{C}, m)$.

ii.) If $(\mathscr{C}, m)$ is a weighted polyhedral complex, we also get a canonical weight on every subdivision $\mathscr{D}$ of $\mathscr{C}$ by passing on the m_σ to the maximal polyhedra of $\mathscr{D}$.

iii.) If $(\mathscr{C}, m)$ is a weighted polyhedral complex, there is a subcomplex

$$\mathscr{D} := \{\Delta \in \mathscr{C} \mid \Delta \subseteq \sigma, \text{ where } \sigma \in \mathscr{C} \text{ is maximal polyhedron with } m_\sigma \neq 0\}.$$

We define the support of the weighted complex $(\mathscr{C}, m)$ as the support of $\mathscr{D}$. We usually neglect the polyhedra of $\mathscr{C} \setminus \mathscr{D}$.

Definition 3.2.5. Let $(\mathscr{C}, m)$ be a weighted integral $\mathbb{R}$-affine polyhedral complex of pure dimension n. For $\alpha \in A^{n,n}_{\mathscr{C},c}(\Omega)$ we define the integral

$$\int_{(\mathscr{C},m)} \alpha := \sum_{\sigma \in \mathscr{C}_n} m_\sigma \int_\sigma \alpha,$$

where on the right hand side we integrate as in Definition 3.1.14. Analogously we define integration along the boundary of $\mathscr{C}$ for $\beta \in A^{n-1,n}_{\mathscr{C},c}(\Omega)$ resp. $A^{n,n-1}_{\mathscr{C},c}(\Omega)$ by

$$\int_{\partial(\mathscr{C},m)} \beta := \sum_{\sigma \in \mathscr{C}_n} m_\sigma \int_{\partial\sigma} \beta,$$

where on the right we integrate along boundaries as in Definition 3.1.16.

With Stokes' formula for polyhedra 3.1.18 we immediately get the following version for polyhedral complexes.

Proposition 3.2.6. (Stokes' Formula) Let $(\mathscr{C}, m)$ be a weighted integral $\mathbb{R}$-affine polyhedral complex of pure dimension n and Ω an open subset of $|\mathscr{C}|$. For any $\eta' \in A^{n-1,n}_{\mathscr{C},c}(\Omega)$ and any $\eta'' \in A^{n,n-1}_{\mathscr{C},c}(\Omega)$, we have

$$\int_{(\mathscr{C},m)} d'\eta' = \int_{\partial(\mathscr{C},m)} \eta', \quad \int_{(\mathscr{C},m)} d''\eta'' = \int_{\partial(\mathscr{C},m)} \eta''.$$

Definition 3.2.7. A weighted integral $\mathbb{R}$-affine polyhedral complex $(\mathscr{C}, m)$ of pure dimension n is a *tropical cycle of dimension* n if its weight m satisfies the *balancing condition*, i.e. for every $(n-1)$-dimensional $\rho \in \mathscr{C}$ we have

$$\sum_{\sigma \in \mathscr{C}_n,\ \rho \prec \sigma} m_\sigma \omega_{\rho,\sigma} \in \mathbb{Z}_\rho.$$

Remark 3.2.8. If $(\mathscr{C}, m)$ is a weighted integral $\mathbb{R}$-affine polyhedral complex of pure dimension n, then we obtain a supercurrent $\delta_{(\mathscr{C},m)} \in D_{n,n}(\mathbb{R}^r)$ by setting $\langle \delta_{(\mathscr{C},m)}, \eta \rangle = \int_{(\mathscr{C},m)} \eta$ for any $\eta \in A^{n,n}_c(\mathbb{R}^r)$. By [Gub16, Proposition 3.8] the following conditions are equivalent:

i.) $(\mathscr{C}, m)$ is a tropical cycle;

ii.) $\delta_{(\mathscr{C},m)}$ is a d'-closed supercurrent on $\mathbb{R}^r$;

iii.) $\delta_{(\mathscr{C},m)}$ is a d''-closed supercurrent on $\mathbb{R}^r$.

3.3 Moment maps and tropical charts

In this section we introduce the notion of tropical charts, with which we will be able to define differential forms on the Berkovich analytification of algebraic varieties.

From now on, K is an algebraically closed field endowed with a complete nontrivial non-Archimedean absolute value $|\cdot|_K$ (sometimes we just write $|\cdot|$, there is no ambiguity). In particular the residue field $\widetilde{K}$ is also algebraically closed. Let $\nu := -\log|\cdot|$ be the associated valuation and $\Gamma := \nu(K^*) \subseteq \mathbb{R}$ its value group. Note that Γ is a divisible, dense, additive subgroup of $\mathbb{R}$.

Also in the following let X always be an algebraic variety over K, i.e. an integral, separated K-scheme of finite type. Recall that a scheme is integral if and only if it is reduced and irreducible. Any open subscheme of X is an algebraic variety again.

In the following for the fiber product of algebraic varieties X and Y (if not stated otherwise) we always consider the scheme theoretic fiber product over $\mathrm{Spec}(K)$, i.e. we write $X \times Y$ for $X \times_{\mathrm{Spec}(K)} Y$.

In this section we will very often use the following basic fact from algebraic geometry which we want to call to mind: Let $(Y, \mathcal{O}_Y)$ be a locally K-ringed space and A a K-algebra. Then there is a natural bijection

$$\begin{aligned} \mathrm{Hom}_{\mathrm{LRS}/K}((X,\mathcal{O}_X),(\mathrm{Spec}(A),\mathcal{O}_{\mathrm{Spec}(A)})) &\xrightarrow{\sim} \mathrm{Hom}_{K\text{-alg.}}(A,\mathcal{O}_X(X)) \\ (F,F^{\#}) &\mapsto F^{\#}(\mathrm{Spec}(A)), \end{aligned}$$

where the left hand side denotes the morphisms of locally K-ringed spaces, and the right hand side the morphisms of K-algebras.

Remark 3.3.1. (Berkovich Analytification) We want to briefly recall the most important properties of the Berkovich analytification of the algebraic variety X for our purposes. For an elaborate description we refer to [Ber90, Chapter 3].

The topological space of the analytification X^{an} of X is the space of all pairs $(\mathfrak{p}, p = |\cdot|_p)$, where $\mathfrak{p} \in X$ and $|\cdot|_p$ is an absolute value on the field extension $k(\mathfrak{p}) = \mathcal{O}_{X,\mathfrak{p}}/\mathfrak{m}_{X,\mathfrak{p}}$ over K, which induces the non-Archimedean absolute value $|\cdot|_K$ on K. The space X^{an} is endowed with the coarsest topology such that the map

$$\pi := \ker\colon X^{\mathrm{an}} \to X, \quad (\mathfrak{p}, p = |\cdot|_p) \mapsto \mathfrak{p}$$

is continuous and such that for each Zariski open subset U in X and each $f \in \mathcal{O}_X(U)$ the map

$$\pi^{-1}(U) \to \mathbb{R}_{\geq 0}, \quad (\mathfrak{p}, p = |\cdot|_p) \mapsto |f(p)| := |f(\mathfrak{p})|_p$$

is continuous.

Furthermore note that if $X = \mathrm{Spec}(A)$ is affine, we can view X^{an} as the space of multiplicative seminorms on A extending $|\cdot|_K$ endowed with the coarsest topology such that for any $a \in A$ the map

$$|a|\colon X^{\mathrm{an}} \to \mathbb{R}_{\geq 0}, \quad p = |\cdot|_p \mapsto |a|_p =: |a(p)|$$

is continuous.

The space X^{an} comes with a sheaf $\mathcal{O}_{X^{\mathrm{an}}}$ of *analytic functions* which turns X^{an} into a locally K-ringed space. We will not go into detail here, however the continuous map $\pi\colon X^{\mathrm{an}} \to X$ on topological spaces extends to a morphism of locally K-ringed spaces. For any $p \in X^{\mathrm{an}}$, its induced map on stalks

$$\mathcal{O}_{X,\pi(p)} \to \mathcal{O}_{X^{\mathrm{an}},p}$$

is local and flat, thus faithfully flat and in particular injective. Hence π induces an injection

$$\mathcal{O}_X(U) \hookrightarrow \mathcal{O}_{X^{\mathrm{an}}}(U^{\mathrm{an}})$$

for any open $U \subseteq X$.

A morphism of varieties $\varphi\colon X \to Y$ yields a morphism on the analytifications $\varphi^{\mathrm{an}}\colon X^{\mathrm{an}} \to Y^{\mathrm{an}}$, which preserves immersions, injections, surjections, finiteness and separatedness. On topological spaces, φ^{an} is given locally by precomposing with $\varphi^{\#}$, i.e. if $\mathfrak{q} = \varphi(\mathfrak{p})$ for some $\mathfrak{p} \in X$, a pair $(\mathfrak{p}, |\cdot|_p)$ maps to $(\mathfrak{q}, |\cdot|_q)$, where $|\cdot|_q$ is obtained via the composition

$$\mathcal{O}_{Y,\mathfrak{q}}/\mathfrak{m}_{Y,\mathfrak{q}} \xrightarrow{\varphi^{\#}_{\mathfrak{p}}} \mathcal{O}_{X,\mathfrak{p}}/\mathfrak{m}_{X,\mathfrak{p}} \xrightarrow{|\cdot|_p} \mathbb{R}_{\geq 0}.$$

In the affine case, for a morphism $\varphi\colon X = \mathrm{Spec}(B) \to Y = \mathrm{Spec}(A)$ with $\varphi = \mathrm{Spec}(f\colon A \to B)$, the space X^{an} (respectively Y^{an}) can be seen as the set of multiplicative seminorms on B (respectively A) extending $|\cdot|_K$ and on topological spaces we have

$$\begin{aligned} \varphi^{\mathrm{an}}\colon X^{\mathrm{an}} &\to Y^{\mathrm{an}}, \\ |\cdot|_p &\mapsto [a \mapsto |f(a)|_p]. \end{aligned} \tag{3.3.1}$$

For any variety X, the space X^{an} is locally compact and Hausdorff as X is separated. Furthermore X^{an} is compact if and only if X is proper (see [Ber90, Theorem 3.4.8]).

Finally, for the variety X and every good K-analytic space Y and any morphism of locally K-ringed spaces $\varphi\colon Y \to X$ there exists a unique morphism of K-analytic spaces $\varphi'\colon Y \to X^{\mathrm{an}}$ such that $\varphi = \pi \circ \varphi'$.

For $p \in X^{\mathrm{an}}$, denote by $\mathcal{H}(p)$ the completion of the residue field $k(\pi(p))$ with respect to $|\cdot|_p$. For a K-affinoid algebra A we denote the Berkovich spectrum by $\mathcal{M}(A)$.

Remark 3.3.2. As in [Ber90, §9.1], for a point $x \in X^{\mathrm{an}}$ we associate nonnegative integers $s(x)$ and $t(x)$ by

$$\begin{aligned} s(x) &:= \operatorname{trdeg}(\widetilde{\mathcal{H}(x)}/\widetilde{K}),\\ t(x) &:= \dim_{\mathbb{Q}} \mathbb{Q} \otimes_{\mathbb{Z}} \left(\sqrt{|\mathcal{H}(x)^*|}/\sqrt{|K^*|}\right). \end{aligned}$$

and define $d(x) := s(x) + t(x)$. By Abhyankar's inequality $d(x) \leq \operatorname{trdeg}(\mathcal{H}(x)/K)$. Note that by [Ber90, Prop. 9.1.3] we have

$$\dim(X) = \dim_K(V) = \sup_{x \in V} d(x)$$

for any open subset $V \subseteq X^{\mathrm{an}}$. Here $\dim(X)$ denotes the dimension of the scheme X, and $\dim_K(V)$ is the K-analytic dimension of $V \subseteq X^{\mathrm{an}}$. For a detailed treatment of K-analytic dimension see [Duc07, Définition 1.13].

Definition 3.3.3. We write $T = \mathbb{G}_m^r = \operatorname{Spec}(K[T_1^{\pm 1}, \ldots, T_r^{\pm 1}])$ for the split multiplicative torus of rank r with coordinates $T_1, \ldots, T_r$. Recall that T is an affine algebraic variety via

$$T \cong \operatorname{Spec}(K[T_1, \ldots, T_r, S_1, \ldots, S_r]/(T_1 S_1 - 1, \ldots, T_r S_r - 1)).$$

i.) We define the *tropicalization map*

$$\operatorname{trop}\colon T^{\mathrm{an}} \to \mathbb{R}^r, \quad p \mapsto (-\log|T_1(p)|, \ldots, -\log|T_r(p)|).$$

ii.) For a closed subvariety Y of T, we call $\operatorname{Trop}(Y) := \operatorname{trop}(Y^{\mathrm{an}})$ the *tropical variety associated with* Y.

Lemma 3.3.4. Let $T = \mathbb{G}_m^q$. The tropicalization map $\operatorname{trop}\colon T^{\mathrm{an}} \to \mathbb{R}^q$ is a continuous and proper map of topological spaces.

Proof. Continuity is clear by construction of the topology on analytifications. To see that trop is proper, it suffices to show that for any $a_i < b_i$ the set

$$V := \{p \in T^{\mathrm{an}} \mid a_i \leq |T_i(p)| \leq b_i,\ i = 1, \dots, q\}$$

is compact. A K-analytic atlas on T^{an} is given by charts of the form

$$(\mathcal{D}(0,r), \mathcal{O}_{\mathcal{M}(K_r)|\mathcal{D}(0,r)})$$

where $r = (r_1, \dots, r_q) \in \mathbb{R}^q_{>0}$ and the annulus $K_r = K\{r_i^{-1}T_i, r_iS_i\}/(S_iT_i - 1)$ and $\mathcal{D}(0,r) = \{p \in \mathcal{M}(K_r) \mid |T_i(p)| < r_i\ \forall i\}$. Hence V is a Laurent domain in a chart with large enough r_i's, in particular compact. □

Remark 3.3.5. For a closed subvariety Y of T of dimension n, the *Bieri-Groves-Theorem* (see [Gub13, Theorem 3.3]) states that $\mathrm{Trop}(Y)$ is a finite union of n-dimensional integral Γ-affine polyhedra in $\mathbb{R}^r$. In tropical geometry it is shown even further that the polyhedra can be chosen such that $\mathrm{Trop}(Y)$ is an integral Γ-affine polyhedral complex. The structure of this complex is only determined up to subdivision, which does not matter for our constructions though.

Remark 3.3.6. If $Y \subseteq T$ is a closed subvariety as above, it is (nontrivially) possible to define a *tropical weight* m on an n-dimensional polyhedron σ of $\mathrm{Trop}(Y)$, such that $(\mathrm{Trop}(Y), m)$ is a tropical cycle. For details we refer to [Gub13, Section 13].

Example 3.3.7. Let $f = \sum_{c_u \in \mathbb{Z}^n} c_u T^u \in K[T_1^{\pm 1}, \dots, T_n^{\pm 1}]$ be an irreducible Laurent polynomial and we consider the closed affine subvariety

$$X := \mathrm{Spec}(K[T_1^{\pm 1}, \dots, T_n^{\pm 1}]/(f))$$

of $\mathbb{G}_m^n$. Recall that we have defined $\nu := -\log|\cdot|_K$. In tropical geometry (see e.g. [MS15, §3.1]), the *tropicalization* $\mathrm{trop}(f)$ of the Laurent polynomial f is defined as the piecewise affine function

$$\mathrm{trop}(f)\colon \mathbb{R}^n \to \mathbb{R},$$
$$w \mapsto \min\{\nu(c_u) + \langle w, u\rangle \mid u \in \mathbb{Z}^n \text{ and } c_u \neq 0\}$$

and the *tropical hypersurface* $\mathrm{trop}(V(f))$ as the set

$$\{w \in \mathbb{R}^n \mid \text{the minimum in } \mathrm{trop}(f)(w) \text{ is achieved at least twice}\}, \tag{3.3.2}$$

which is the locus where $\mathrm{trop}(f)$ fails to be affine.

Naturally the question arises how $\operatorname{trop}(V(f))$ fits with the notion of $\operatorname{Trop}(X)$ as in Definition 3.3.3:

For $x \in X$ we write as usual $k(x) = \mathcal{O}_{X,x}/\mathfrak{m}_{X,x}$ for the corresponding residue field. For every K-rational point $x \in X(K)$ of the variety X, we have an isomorphism $k(x) \xrightarrow{\sim} K$ and thus obtain a natural embedding $X(K) \hookrightarrow X^{\mathrm{an}}$. By [Gub13, 2.6] the set $X(K)$ is dense in X^{an} and by continuity of the tropicalization map the tropical variety $\operatorname{Trop}(X) = \operatorname{trop}(X^{\mathrm{an}})$ is equal to the closure of the set

$$\{(-\log|T_1(p)|, \ldots, -\log|T_n(p)|) \mid p \in X(K) \subseteq X^{\mathrm{an}}\} \subseteq \mathbb{R}^n. \tag{3.3.3}$$

We have the canonical identification of $X(K)$ with the roots of f

$$X(K) = \{a \in (K^*)^n \mid f(a) = 0\} =: V(f).$$

Each $a \in X(K)$ corresponds to the multiplicative seminorm $|f|_a = |f(a)|_K$ in X^{an}. By (3.3.3) we obtain

$$\operatorname{Trop}(X) = \overline{\{(-\log|a_1|, \ldots, -\log|a_n|) \in \mathbb{R}^n \mid a = (a_1, \ldots, a_n) \in V(f)\}}.$$

Now Kapranov's Theorem (see [MS15, Theorem 3.1.3]) then tells us that the tropical hypersurface $\operatorname{trop}(V(f))$ as defined in (3.3.2) is equal to the closure of

$$\{(-\log|a_1|, \ldots, -\log|a_n|) \in \mathbb{R}^n \mid a = (a_1, \ldots, a_n) \in V(f)\} \subseteq \mathbb{R}^n.$$

In particular we get the equality

$$\operatorname{Trop}(X) = \operatorname{trop}(V(f)).$$

This equality can significantly ease explicit computations. For example consider the algebraically closed field of Puiseux series

$$\mathbb{C}\{\{t\}\} = \bigcup_{n \in \mathbb{N}_{\geq 1}} \mathbb{C}(\!(t^{1/n})\!),$$

where $\mathbb{C}(\!(t^{1/n})\!)$ is the field of formal Laurent series in the variable $t^{1/n}$. This field carries a natural nontrivial non-Archimedean absolute value by setting $|f(t)| := \exp(-\operatorname{val}(f(t)))$ for any $f(t) \in \mathbb{C}\{\{t\}\}^*$, where $\operatorname{val}(f(t))$ is the lowest exponent i/n that appears in the series expansion of $f(t)$. Finally let K be the completion of $\mathbb{C}\{\{t\}\}$ with respect to $|\cdot|$ and we consider the bivariate polynomial $f \in K[X^{\pm 1}, Y^{\pm 1}]$ given by $f = tX^3Y^2 + t^2XY + (1+t^4)X$.

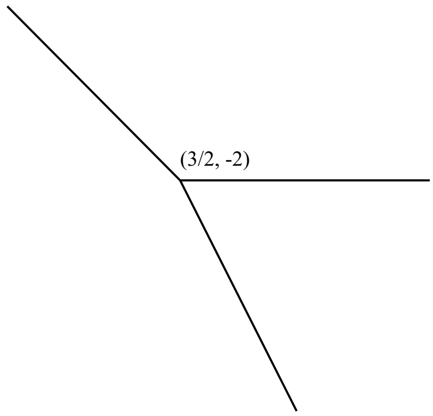

FIGURE 3.1: Tropical curve of $tX^3Y^2 + t^2XY + (1 + t^4)X$.

Then the tropicalization of f is given by

$$\operatorname{trop}(f)(w_1, w_2) = \min\{1 + 3w_1 + 2w_2, 2 + w_1 + w_2, w_1\},$$

and one easily sees that

$$\begin{aligned}\operatorname{Trop}(\operatorname{Spec}(K[X^{\pm 1}, Y^{\pm 1}]/(f))) = \operatorname{trop}(V(f)) &= \{w_2 = 1 - 2w_1 \wedge w_1 \geq \frac{3}{2}\} \\ &\cup \{w_2 = -\frac{1}{2} - w_1 \wedge w_1 \leq \frac{3}{2}\} \\ &\cup \{w_2 = -2 \wedge w_1 \geq \frac{3}{2}\}.\end{aligned}$$

The corresponding polyhedral complex is the tropical curve is sketched in Figure 3.1.

Definition and Remark 3.3.8. Let U be an open subset of the algebraic variety X.

i.) A *moment map* is a scheme morphism $\varphi\colon U \to T$ to some split multiplicative torus $T = \mathbb{G}_m^r$. The *tropicalization of* φ is defined as

$$\varphi_{\mathrm{trop}} := \operatorname{trop} \circ \varphi^{\mathrm{an}} \colon U^{\mathrm{an}} \xrightarrow{\varphi^{\mathrm{an}}} T^{\mathrm{an}} \xrightarrow{\operatorname{trop}} \mathbb{R}^r.$$

This is a continuous map with respect to the topology on U^{an}.

Let $U' \subseteq U$ be another open subset with moment map $\varphi'\colon U' \to T' = \mathbb{G}_m^{r'}$. We say that φ' *refines* φ if there exists an affine morphism of tori $\psi\colon \mathbb{G}_m^{r'} \to \mathbb{G}_m^r$, such that $\varphi = \psi \circ \varphi'$ on U'.

Here an *affine morphism of tori* stems from a group homomorphism $\mathbb{Z}^r \to \mathbb{Z}^{r'}$ composed with a (multiplicative) translation, i.e. it comes from a morphism of K-algebras

$$K[T_1^{\pm 1}, \ldots, T_r^{\pm 1}] \to K[T_1^{\pm 1}, \ldots, T_{r'}^{\pm 1}]$$
$$T_i \mapsto a_i T^{z_i},$$

where $a_i \in K^*$ and $z_i = (z_{i,1}, \ldots, z_{i,r'}) \in \mathbb{Z}^{r'}$ with $T^{z_i} := T_1^{z_{i,1}} \cdots T_{r'}^{z_{i,r'}}$.

ii.) Now in the situation of a refinement with the notation as above, let $x \in (U')^{\mathrm{an}} \subseteq U^{\mathrm{an}}$ and set $c := \varphi^{\mathrm{an}}(x) \in T^{\mathrm{an}}$ respectively $c' := (\varphi')^{\mathrm{an}}(x) \in (T')^{\mathrm{an}}$. The i-th component of $\varphi_{\mathrm{trop}}(x) \in \mathbb{R}^r$ satisfies

$$\varphi_{\mathrm{trop}}(x)_i = -\log|T_i(c)| = -\log|T_i(\psi^{\mathrm{an}}(c'))| \overset{(3.3.1)}{=} -\log|\psi(T_i)(c')| =$$
$$= -\log|a_i T^{z_i}(c')| = -\log|a_i| + \sum_{j=1}^{r'} z_{i,j}\left(-\log|T_j(c')|\right) = -\underbrace{\log|a_i|}_{\in\Gamma} + \sum_{j=1}^{r'} z_{i,j}\varphi'_{\mathrm{trop}}(x)_j.$$

If we write Z for the matrix $(z_{i,j})_{ij} \in M(r \times r', \mathbb{Z})$ and $a := (-\log|a_i|)_{i=1,\ldots,r} \in \Gamma^r$, then with the above we see that ψ induces an integral Γ-affine map

$$\mathrm{Trop}(\psi)\colon \mathbb{R}^{r'} \to \mathbb{R}^r,\ y \mapsto Zy + a,$$

such that $\varphi_{\mathrm{trop}} = \mathrm{Trop}(\psi) \circ \varphi'_{\mathrm{trop}}$ on $(U')^{\mathrm{an}}$.

Remark 3.3.9. If $\varphi_i\colon U_i \to \mathbb{G}_m^{r_i}$ are finitely many moment maps of nonempty open subsets $U_1, \ldots, U_n$ of X, then $U := \bigcap_i U_i$ is an open subset of X which is nonempty (as X is irreducible). Note that the fiber product

$$\prod_i \mathbb{G}_m^{r_i} \cong \mathrm{Spec}(\bigotimes_i K[T_1^{\pm 1}, \ldots, T_{r_i}^{\pm 1}]) \cong \mathbb{G}_m^{\sum_i r_i}$$

is a split torus as well and the universal property of the fiber product yields a morphism

$$\varphi := \varphi_1 \times \cdots \times \varphi_n\colon U \to \mathbb{G}_m^{\sum_i r_i},$$

which refines each φ_i via the canonical projection maps (these are clearly affine morphism of tori). Moreover the universal property of the fiber product immediately yields that for $U' \subseteq U$ open every moment map $\varphi'\colon U' \to T'$ which refines every φ_i also refines φ.

Lemma 3.3.10. Let $\varphi\colon U \to \mathbb{G}_m^r$ be a moment map on an open subset U of X and let U' be a nonempty open subset of U. Then $\varphi_{\mathrm{trop}}((U')^{\mathrm{an}}) = \varphi_{\mathrm{trop}}(U^{\mathrm{an}})$.

Proof. See [Gub16, Lemma 4.9].

Remark 3.3.11. Let $U \subseteq X$ be an open affine subset. We construct a *canonical moment map* φ_U as follows: By an analogy of Dirichlet's unit theorem the group $M_U := \mathcal{O}_X(U)^*/K^*$ is free of finite rank. Choose representatives $\varphi_1, \dots, \varphi_r \in \mathcal{O}_X(U)^*$ of a basis, and we obtain a K-algebra morphism

$$K[T_1^{\pm 1}, \dots, T_r^{\pm 1}] \to \mathcal{O}_X(U),$$
$$T_i \mapsto \varphi_i$$

which gives a moment map $\varphi_U \colon U \to \mathbb{G}_m^r =: T_U$. Note that this moment map is 'canonical' up to base change and multiplicative translation by elements of K^*.

Remark 3.3.12. Let $f \colon X' \to X$ be a morphism of algebraic varieties over K and let $U' \subseteq X'$ and $U \subseteq X$ be open subsets with $f(U') \subseteq U$. Denote by g the composition of ring morphisms

$$\mathcal{O}_X(U) \xrightarrow{f^\#(U)} \mathcal{O}_{X'}(\underbrace{f^{-1}(U)}_{\supseteq U'}) \xrightarrow{|_{U'}} \mathcal{O}_{X'}(U').$$

If $\varphi_1, \dots, \varphi_r \in \mathcal{O}_X(U)^*$ (respectively $\varphi'_1, \dots, \varphi'_{r'}$) are lifts of a basis of M_U (respectively $M_{U'}$), then $g(\varphi_i) = a_i(\varphi')^{z_i}$ for $a_i \in K^*$ and $z_i \in \mathbb{Z}^{r'}$, where we write as usual $(\varphi')^{z_i} := (\varphi'_1)^{z_{i,1}} \cdots (\varphi'_{r'})^{z_{i,r'}}$. The K-algebra morphism

$$K[T_1^{\pm 1}, \dots, T_r^{\pm 1}] \to K[T_1^{\pm 1}, \dots, T_{r'}^{\pm 1}],$$
$$T_i \mapsto a_i T^{z_i}$$

induces an affine morphism of tori $\psi_{U,U'} \colon \mathbb{G}_m^{r'} \to \mathbb{G}_m^r$, satisfying

$$\psi_{U,U'} \circ \varphi_{U'} = \varphi_U \circ f$$

on U'. To see the equality note again that $\mathbb{G}_m^r$ is affine, hence

$$\mathrm{Hom}_{\mathrm{LRS}}(U, \mathbb{G}_m^r) \cong \mathrm{Hom}_{K\text{-alg.}}(K[T_1^{\pm 1}, \dots, T_r^{\pm 1}], \mathcal{O}_X(U)),$$

so it suffices to easily check the equality on the images of T_i under the corresponding ring map.

In the case $X = X'$ and $f = \mathrm{id}$, we obtain an affine morphism of tori $\psi_{U,U'} \colon \mathbb{G}_m^{r'} \to \mathbb{G}_m^r$ such that $\psi_{U,U'} \circ \varphi_{U'} = \varphi_U$ on U'. Hence for an inclusion $U' \subseteq U$ of open subsets in X, the canonical moment map $\varphi_{U'}$ always refines φ_U in the sense of Definition 3.3.8.

Definition 3.3.13. An open subset $U \subseteq X$ is called *very affine* if U allows a closed immersion into some multiplicative split torus.

Remark 3.3.14. For an open affine subset $U \subseteq X$ the following properties are clearly equivalent:

i.) The canonical moment map φ_U is a closed embedding.

ii.) U is very affine.

iii.) $\mathcal{O}_X(U)$ is finitely generated as a K-algebra by $\mathcal{O}_X(U)^*$.

The following lemma shows that all local considerations can be done using very affine open subsets.

Lemma 3.3.15. Let X be an algebraic variety.

i.) The intersection of two very affine subsets $U \hookrightarrow \mathbb{G}_m^r, U' \hookrightarrow \mathbb{G}_m^{r'}$ of X is very affine again.

ii.) The very affine open subsets of X form a basis for the Zariski topology on X.

Proof. i.) As X is separated, the intersection of two affine subsets $U \cap U'$ is affine again and the canonical map $U \cap U' \to U \times U'$ is a closed immersion. The natural map $\varphi \times \varphi' \colon U \times U' \to \mathbb{G}_m^r \times \mathbb{G}_m^{r'} \cong \mathbb{G}_m^{r+r'}$ is also a closed immersion, as the corresponding map on the tensor products is surjective. Hence $U \cap U' \to \mathbb{G}_m^{r+r'}$ is a closed immersion.

ii.) Let $x \in X$ and $U \subseteq X$ be an open neighborhood of x. It suffices to show that there is a very affine open V around x with $V \subseteq U$. In particular we can assume that U is affine with $U = \operatorname{Spec}(A)$, where A is an integral K-algebra of finite type, i.e. it is of the form $A = K[T_1, \dots, T_n]/\mathfrak{a}$ for some ideal $\mathfrak{a}$. Let $\mathfrak{p}$ denote the prime ideal of A corresponding to x and $\overline{T}_i$ the class of T_i in A. Consider the elements $f_1, \dots, f_n \in A$ with

$$f_i = \begin{cases} \overline{T_i}, \text{ if } \overline{T_i} \notin \mathfrak{p} \\ \overline{T_i} + 1, \text{ if } \overline{T_i} \in \mathfrak{p} \end{cases} .$$

Then $V := D(f_1) \cap \dots \cap D(f_n) = D(f_1 \cdots f_n) \subseteq U = \operatorname{Spec}(A)$ is open around x and corresponds to the localization $A\left[\frac{1}{f_1 \cdots f_n}\right]$. We obtain a surjective K-algebra morphism

$$K[T_1^{\pm 1}, \dots, T_n^{\pm 1}] \to A\left[\frac{1}{f_1 \cdots f_n}\right]$$
$$T_i \to f_i$$

which gives a closed immersion $V \hookrightarrow \mathbb{G}_m^n$.

□

Remark 3.3.16. On a very affine open subset, we will always use the canonical moment map $\varphi_U \colon U \to T_U := \mathbb{G}_m^r$, which is a closed immersion by Remark 3.3.14. We write $\mathrm{Trop}(U) := \mathrm{Trop}(\varphi_U(U)) \subseteq \mathbb{R}^r$ for the tropical variety of U in T_U, which is a tropical cycle. For the tropicalization map we briefly write $\mathrm{trop}_U := (\varphi_U)_{\mathrm{trop}} \colon U^{\mathrm{an}} \to \mathbb{R}^r$. Recall that φ_U is only determined up to multiplicative translation and change of basis. Hence by Definition 3.3.8, trop_U and $\mathrm{Trop}(U)$ are only canonical up to affine transformation.

Definition 3.3.17.

i.) A *tropical chart* (V, φ_U) on X^{an} consists of an open subset V of X^{an} contained in U^{an} for a very affine open subset U of X with $V = \mathrm{trop}_U^{-1}(\Omega)$ for some open subset Ω of $\mathrm{Trop}(U)$. Observe that in this case $\mathrm{trop}_U(V) = \Omega$.

ii.) A tropical chart $(V', \varphi_{U'})$ is called a *tropical subchart* of (V, φ_U) if $V' \subseteq V$ and $U' \subseteq U$.

Remark 3.3.18. Recall again that the analytification of morphisms preserves immersions, i.e. if $U \subseteq U'$ as open subvarieties, then $U^{\mathrm{an}} \subseteq (U')^{\mathrm{an}}$. Hence it makes sense to talk about inclusions $(U')^{\mathrm{an}} \subseteq U^{\mathrm{an}} \subseteq X^{\mathrm{an}}$ as in the definition above and about intersections as below.

Remark 3.3.19. Let $(V', \varphi_{U'})$ be a tropical subchart of (V, φ_U) with $V' = \mathrm{trop}_{U'}^{-1}(\Omega')$ respectively $V = \mathrm{trop}_U^{-1}(\Omega)$ as above. By Remark 3.3.12, $\varphi_{U'}$ refines φ_U and there exists an affine morphism of tori $\psi_{U,U'}$ such that $\psi_{U,U'} \circ \varphi_{U'} = \varphi_U$ on U' and hence $\mathrm{trop}_U = \mathrm{Trop}(\psi_{U,U'}) \circ \mathrm{trop}_{U'}$ on $(U')^{\mathrm{an}}$, where $\mathrm{Trop}(\psi_{U,U'})$ is defined as in Remark 3.3.8 ii.).

i.) We obtain

$$\begin{aligned} \mathrm{Trop}(U) = \mathrm{trop}_U(U^{\mathrm{an}}) &\overset{3.3.10}{=} \mathrm{trop}_U((U')^{\mathrm{an}}) = \\ = (\mathrm{Trop}(\psi_{U,U'}) \circ \mathrm{trop}_{U'})((U')^{\mathrm{an}}) &= \mathrm{Trop}(\psi_{U,U'})(\mathrm{Trop}(U')). \end{aligned}$$

Hence the integral Γ-affine map $\mathrm{Trop}(\psi_{U,U'}) \colon \mathbb{R}^{r'} \to \mathbb{R}^r$ restricts to a surjective affine map of supports of polyhedral complexes

$$\mathrm{Trop}(\psi_{U,U'}) \colon \mathrm{Trop}(U') \to \mathrm{Trop}(U).$$

Furthermore this yields

$$\mathrm{Trop}(\psi_{U,U'})(\Omega') = \mathrm{Trop}(\psi_{U,U'})(\mathrm{trop}_{U'}(V')) = \mathrm{trop}_U(\underbrace{V'}_{\subseteq V}) \subseteq \Omega.$$

ii.) If $\sigma \in \mathrm{Trop}(U)$ and $\sigma' \in \mathrm{Trop}(U')$ are polyhedra with $\mathrm{Trop}(\psi_{U,U'})(\sigma') = \sigma$, then by i.) the induced linear map $\mathbb{L}(\sigma') \to \mathbb{L}(\sigma)$ of underlying vector spaces is surjective, and by the Open Mapping Theorem hence open. In particular

$$\mathrm{Trop}(\psi_{U,U'})(\Omega' \cap \sigma') \subseteq \Omega \cap \sigma$$

is open in σ.

Proposition 3.3.20. The tropical charts on X^{an} have the following properties:

i.) For every open subset $W \subseteq X^{\mathrm{an}}$ and every $x \in W$ there exists a tropical chart (V, φ_U) with $x \in V \subseteq W$. Furthermore, V can be chosen such that the open subset $\mathrm{trop}_U(V) \subseteq \mathrm{Trop}(U)$ is relatively compact.

ii.) The intersection $(V \cap V', \varphi_{U \cap U'})$ of tropical charts (V, φ_U) and $(V', \varphi_{U'})$ is a tropical subchart of both.

iii.) If (V, φ_U) is a tropical chart and if U'' is a very affine open subset of U with $V \subseteq (U'')^{\mathrm{an}}$, then $(V, \varphi_{U''})$ is a tropical subchart of (V, φ_U).

iv.) Let $V \subseteq U^{\mathrm{an}}$ be an open subset where $U \subseteq X$ is a very affine open, and let $x \in V$. Then there exists a tropical chart $(V', \varphi_{U'})$ with $x \in V' \subseteq V$ and $U' \subseteq U$.

Proof. i.) As the very affine open subsets form a basis of the Zariski topology on X and X^{an} can be obtained by glueing the analytifications of its affine opens, we may assume that $X = \mathrm{Spec}(A)$ is a very affine variety. A basis of X^{an} is formed by subsets of the form $V := \{x \in X^{\mathrm{an}} \mid s_1 < |f_1(x)| < r_1, \ldots, s_k < |f_k(x)| < r_k\}$ with all $f_i \in A$ and real numbers $s_i < r_i$. We can even assume that all $s_i > 0$. Indeed, let $r > 0$. As $|K^*|$ lies dense in $\mathbb{R}_{\geq 0}$, we can find a sequence $(a_n)_{n\in\mathbb{N}}$ in K^*, such that $\lim_{n\to\infty} |a_n|_K = 0$ and all $|a_n|_K < r$, and it is easy to check using the ultrametric triangle inequality that

$$\{x \in X^{\mathrm{an}} \mid |f(x)| \in [0, r)\} = \bigcup_{i\in\mathbb{N}} \{x \in X^{\mathrm{an}} \mid |(f + a_i)(x)| \in (\frac{1}{2}\underbrace{|a_i|_K}_{>0}, r)\}$$

for any $f \in A$.

Now any V of such a form lies in the analytification of the nonvanishing locus $U := \{x \in X \mid f_1(x) \neq 0, \ldots, f_k(x) \neq 0\}$, which clearly is very affine again as localization at $f_1, \ldots, f_k$ of the very affine variety X. In order to show that (V, φ_U) is a tropical chart, it remains to show that $V = \mathrm{trop}_U^{-1}(\Omega)$ for some open subset Ω of $\mathrm{Trop}(U)$.
For this, let $g_1, \ldots, g_n \in \mathcal{O}_X(U)^* = A[\frac{1}{f_1 \cdots f_k}]^*$ be lifts of a basis of $\mathcal{O}_X(U)^*/K^*$. We can assume without loss of generality that φ_U is given by the map

$$\psi : K[T_1^{\pm 1}, \ldots, T_n^{\pm 1}] \to \mathcal{O}_X(U), \quad T_i \mapsto g_i.$$

For any $j \in \{1, \ldots, k\}$, there are $a_j \in K^*$ and $z_j \in \mathbb{Z}^n$, such that $f_j = a_j \cdot g^{z_j} = \psi(a_j T^{z_j})$.

Note that for $x \in U^{\mathrm{an}}$ we have $\mathrm{trop}_U(x) = (-\log(|g_1(x)|), \ldots, -\log(|g_n(x)|))$ and consider for any $j \in \{1, \ldots, k\}$ the continuous map

$$\begin{aligned} \alpha_j \colon \mathbb{R}^n &\to \mathbb{R}, \\ (y_1, \ldots, y_n) &\mapsto |a_j| \cdot \exp\left(-\sum_{i=1}^{n} z_{j,i} y_i\right). \end{aligned}$$

An easy direct computation shows that

$$|f_j(x)| \in (s_j, r_j) \iff \alpha_j \circ \mathrm{trop}_U(x) \in (s_j, r_j)$$

for all $j \in \{1, \ldots, k\}$ and $x \in U^{\mathrm{an}}$.

Hence $V = \mathrm{trop}_U^{-1}\Big(\underbrace{(\bigcap_{j=1}^{k} \alpha_j^{-1}(s_j, r_j)) \cap \mathrm{Trop}(U)}_{=:\Omega \text{ open in } \mathrm{Trop}(U)}\Big)$.

Furthermore $\mathrm{trop}_U(V)$ is relatively compact, as Ω is clearly bounded, hence its closure is compact.

ii.) Let $(V, \varphi_U \colon U \to \mathbb{G}_m^r)$ respectively $(V', \varphi_{U'} \colon U' \to \mathbb{G}_m^{r'})$ be tropical charts with $\Omega = \mathrm{trop}_U(V)$ respectively $\Omega' = \mathrm{trop}_{U'}(V')$ open subsets in $\mathrm{Trop}(U)$ respectively $\mathrm{Trop}(U')$. By Lemma 3.3.15 the intersection $U \cap U'$ is very affine via the closed embedding

$$\Phi \colon U \cap U' \xrightarrow{\alpha} U \times U' \xrightarrow{\varphi_U \times \varphi_{U'}} \mathbb{G}_m^r \times \mathbb{G}_m^{r'} \cong \mathbb{G}_m^{r+r'}.$$

Here α is the closed immersion coming from the canonical surjective map

$$\mathcal{O}_X(U) \otimes_K \mathcal{O}_X(U') \to \mathcal{O}_X(U \cap U').$$

Now consider the following diagram of the underlying topological spaces of analytifications:

$$
\begin{tikzcd}
(U' \cap U)^{\mathrm{an}} \arrow[dr, hook, "\alpha^{\mathrm{an}}"] \arrow[drrr, bend left] \arrow[dddr, bend right] & & & & \\
& (U' \times U)^{\mathrm{an}} \arrow[rr] \arrow[dr, hook, "(\varphi_{U'} \times \varphi_U)^{\mathrm{an}}"] \arrow[dd] & & U^{\mathrm{an}} \arrow[d, "\varphi_U^{\mathrm{an}}"] & \\
& & (\mathbb{G}_m^{r'} \times \mathbb{G}_m^{r})^{\mathrm{an}} \arrow[r, "\pi_r^{\mathrm{an}}"] \arrow[d, "\pi_{r'}^{\mathrm{an}}"'] \arrow[dr, "\mathrm{trop}"'] & (\mathbb{G}_m^{r})^{\mathrm{an}} \arrow[dr, bend left, "\mathrm{trop}"] \arrow[d, phantom, "(1)"] & \\
& (U')^{\mathrm{an}} \arrow[r, "\varphi_{U'}^{\mathrm{an}}"] & (\mathbb{G}_m^{r'})^{\mathrm{an}} \arrow[dr, bend right, "\mathrm{trop}"'] \arrow[r, phantom, "(2)"] & \mathbb{R}^{r'+r} \arrow[r, "\mathrm{proj}"] \arrow[d, "\mathrm{proj}"] & \mathbb{R}^{r} \\
& & & \mathbb{R}^{r'} &
\end{tikzcd}
$$

Here $\mathrm{proj}\colon \mathbb{R}^{r'+r} \to \mathbb{R}^{r'}$ is the projection onto the first r' coordinates (respectively analogously last r coordinates). We want to show that the diagram is commutative. The commutativity follows directly from the universal property of fiber products except for subdiagrams (1) and (2). We show commutativity of (1) (commutativity follows analogously for (2)): Let $x \in (\mathbb{G}_m^{r'} \times \mathbb{G}_m^{r})^{\mathrm{an}}$, which we regard via the isomorphism as in Remark 3.3.9 as a multiplicative seminorm in $\mathrm{Spec}(K[S_1^{\pm 1}, \dots S_{r'}^{\pm 1}, T_1^{\pm 1}, \dots, T_r^{\pm 1}])$. As $\pi_r^{\mathrm{an}}(x)$ is the precomposition of x with

$$K[T_1^{\pm 1}, \dots, T_r^{\pm 1}] \to K[S_1^{\pm 1}, \dots S_{r'}^{\pm 1}] \otimes_K K[T_1^{\pm 1}, \dots, T_r^{\pm 1}] \xrightarrow{\sim} K[S_1^{\pm 1}, \dots S_{r'}^{\pm 1}, T_1^{\pm 1}, \dots, T_r^{\pm 1}],$$

we see that $|T_i(\pi_r^{\mathrm{an}}(x))| = |T_i(x)|$. Then

$$\mathrm{proj} \circ \mathrm{trop}(x) = (-\log |T_1(x)|, \dots, -\log |T_r(x)|) = \mathrm{trop} \circ \pi_r^{\mathrm{an}}(x).$$

Hence the whole diagram commutes. The diagram yields immediately that the set $\Omega'' := \Phi_{\mathrm{trop}}((U \cap U')^{\mathrm{an}}) \cap (\Omega \times \Omega') \subseteq \mathbb{R}^{r+r'}$ is an open subset of $\Phi_{\mathrm{trop}}((U \cap U')^{\mathrm{an}})$. Furthermore by construction of the sets we have

$$\Phi_{\mathrm{trop}}^{-1}(\Omega'') = V \cap V'. \tag{3.3.4}$$

As $\varphi_{U \cap U'}$ refines Φ, we obtain an affine map $\mathrm{Trop}(\psi)$ as in Remark 3.3.8 such that

$$\Phi_{\mathrm{trop}} = \mathrm{Trop}(\psi) \circ \mathrm{trop}_{U \cap U'}$$

on $(U \cap U')^{\mathrm{an}}$ which immediately yields with (3.3.4) that

$$\Omega''' := \mathrm{Trop}(\psi)^{-1}(\Omega'') \cap \mathrm{Trop}(U \cap U')$$

is an open subset of $\mathrm{Trop}(U \cap U')$ with $V \cap V' = \mathrm{trop}_{U\cap U'}^{-1}(\Omega''')$. Thus $(V \cap V', \varphi_{U\cap U'})$ is a tropical chart as well.

iii.) We need to show that $V = \mathrm{trop}_{U''}^{-1}(\Omega'')$ for some open Ω'' in $\mathrm{Trop}(U'')$. As $\varphi_{U''}$ refines φ_U, let as above $\mathrm{Trop}(\psi) := \mathrm{Trop}(\psi_{U,U\cap U''})$ be the affine map with $\mathrm{trop}_U = \mathrm{Trop}(\psi) \circ \mathrm{trop}_{U''}$ on $(U'')^{\mathrm{an}}$. As (V, φ_U) is a tropical chart, let $\Omega := \mathrm{trop}_U(V)$ be the corresponding open subset of $\mathrm{Trop}(U)$. From $V \subseteq (U'')^{\mathrm{an}}$ we get as in ii.) that $V = \mathrm{trop}_{U''}^{-1}(\Omega'')$ for the open subset $\Omega'' := \mathrm{Trop}(\psi)^{-1}(\Omega) \cap \mathrm{Trop}(U'')$.

iv.) Choose tropical chart $(V', \varphi_{\tilde{U}})$ with $x \in V' \subseteq V$ as in i.). Set $U' := \tilde{U} \cap U$, which is very affine again as intersection of very affines. Then iii.) yields that $(V', \varphi_{U'})$ is a tropical chart.

□

3.4 Differential forms on the analytification

Remark 3.4.1. Let us briefly summarize our constructions up to this point. A tropical chart (V, φ_U) consists of an open subset V of U^{an} for a very affine open subset U of X such that $V = \mathrm{trop}_U^{-1}(\Omega)$ for some open subset $\Omega = \mathrm{trop}_U(V)$ of $\mathrm{Trop}(U)$. Here $\varphi_U \colon U \to \mathbb{G}_m^r$ is the canonical moment map. For such a moment map we also shortly write $T_U := \mathbb{G}_m^r$ and $\mathbb{R}_U := \mathbb{R}^r$ (i.e. omit the 'r' to ease notation).

The tropical variety $\mathrm{Trop}(U)$ defines a tropical cycle of $\mathbb{R}_U$ via the tropicalization map $\mathrm{trop}_U \colon U^{\mathrm{an}} \to \mathbb{R}_U$. The canonical map φ_U is only determined up to affine morphism of tori (see Remark 3.3.8), hence all tropical constructions are canonical up to integral Γ-affine isomorphism.

For a tropical subchart $(V', \varphi_{U'}) \subseteq (V, \varphi_U)$ there is an affine morphism of tori

$$\psi_{U,U'} \colon T_{U'} \to T_U$$

with $\varphi_U = \psi_{U,U'} \circ \varphi_{U'}$ on U'. The induced integral Γ-affine map $\mathrm{Trop}(\psi_{U,U'}) \colon \mathbb{R}_{U'} \to \mathbb{R}_U$ surjectively maps the support of the polyhedral complex $\mathrm{Trop}(U')$ onto $\mathrm{Trop}(U)$ (see Remark 3.3.19) with

$$\mathrm{Trop}(\psi_{U,U'})(\mathrm{trop}_{U'}(V')) \subseteq \mathrm{trop}_U(V).$$

Definition 3.4.2. Consider the situation of tropical subcharts $(V', \varphi_{U'}) \subseteq (V, \varphi_U)$ as above. We define the *restriction of a superform* $\alpha \in A^{p,q}_{\mathrm{Trop}(U)}(\Omega)$ on the polyhedral complex $\mathrm{Trop}(U)$ to a superform on $\Omega' := \mathrm{trop}_{U'}(V') \subseteq \mathrm{Trop}(U')$ as the pullback

$$\alpha|_{V'} := \mathrm{Trop}(\psi_{U,U'})^* \alpha \in A^{p,q}_{\mathrm{Trop}(U')}(\Omega').$$

Remark 3.4.3. For tropical subcharts $(\tilde{V}, \varphi_{\tilde{U}}) \subseteq (V', \varphi_{U'}) \subseteq (V, \varphi_U)$ and $\alpha \in A^{p,q}_{\mathrm{Trop}(U)}(\Omega)$, note that

$$\mathrm{Trop}(\psi_{U,\tilde{U}}) = \mathrm{Trop}(\psi_{U,U'}) \circ \mathrm{Trop}(\psi_{U',\tilde{U}}),$$

hence

$$(\alpha|_{V'})|_{\tilde{V}} = \alpha|_{\tilde{V}}.$$

Definition 3.4.4. i.) A *differential form* α of bidegree (p, q) on an open subset V of X^{an} is given by a family $\{(V_i, \varphi_{U_i}, \alpha_i)\}_{i \in I}$ such that the following holds:

a.) For all $i \in I$ the pair (V_i, φ_{U_i}) is a tropical chart of X^{an} and $\bigcup_{i \in I} V_i = V$.

b.) For all $i \in I$ we have $\alpha_i \in A^{p,q}_{\mathrm{Trop}_{U_i}}(\Omega_i)$ with $\Omega_i = \mathrm{trop}_{U_i}(V_i)$.

c.) All α_i agree on intersections, that is for all $(i, j) \in I^2$ we have

$$\alpha_i|_{V_i \cap V_j} = \alpha_j|_{V_i \cap V_j} \in A^{p,q}_{\mathrm{Trop}(U_i \cap U_j)}(\mathrm{trop}_{U_i \cap U_j}(V_i \cap V_j)).$$

ii.) If $\alpha' = \{(V'_i, \varphi_{U'_i}, \alpha'_i)\}_{i \in I'}$ is another differential form on V, then we consider α and α' as the same differential form if and only if

$$\alpha_i|_{V_i \cap V'_j} = \alpha'_j|_{V_i \cap V'_j}$$

for all $(i, j) \in I \times I'$.

iii.) We denote the space of (p, q)-differential forms on V by $A^{p,q}_{X^{\mathrm{an}}}(V)$ (respectively $A^{p,q}(V)$ if the ambient variety X is understood).

iv.) For $\alpha = \{(V_i, \varphi_{U_i}, \alpha_i)\}_{i \in I}$ we define the differential operator

$$d' \colon A^{p,q}(V) \to A^{p+1,q}(V)$$
$$d'\alpha := \{(V_i, \varphi_{U_i}, d'\alpha_i)\}_{i \in I}.$$

Analogously we define d'', d and the wedge product $\wedge$.

Remark 3.4.5. Let $W \subseteq V$ be an inclusion of open subsets in X^{an} and $\alpha = \{(V_i, \varphi_{U_i}, \alpha_i)\}_{i \in I} \in A^{p,q}(V)$. By Proposition 3.3.20 we can choose tropical charts $\{(W_j, \varphi_{U'_j})\}_{j \in J}$ such that $\bigcup_{j \in J} W_j = W$ and for all $j \in J$ there is an $i(j) \in I$ with $W_j \subseteq V_{i(j)}$ and $U'_j \subseteq U_{i(j)}$. We then have a natural restriction

$$\alpha|_W := \{(W_j, \varphi_{U'_j}, \alpha_{i(j)}|_{W_j})\}_{j\in J} \in A^{p,q}(W),$$

which is functorial and well-defined, as it is independent of the choice of tropical charts above.

Indeed, let $\{(\tilde{W}_k, \varphi_{\tilde{U}_k})\}_{k\in K}$ be another cover of W as above and set $\beta := \{(W_j, \varphi_{U'_j}, \alpha_{i(j)}|_{W_j})\}_{j\in J}$ and $\gamma := \{(\tilde{W}_k, \varphi_{\tilde{U}_k}, \alpha_{i(k)}|_{\tilde{W}_k})\}_{k\in K}$. Then for any $(j,k) \in J\times K$ have $W_j \cap \tilde{W}_k \subseteq V_{i(j)} \cap V_{i(k)}$ and hence

$$\begin{aligned}
\beta_j|_{W_j\cap\tilde{W}_k} = (\alpha_{i(j)}|_{W_j})|_{W_j\cap\tilde{W}_k} \qquad &= \alpha_{i(j)}|_{W_j\cap\tilde{W}_k} = (\alpha_{i(j)}|_{V_{i(j)}\cap V_{i(k)}})|_{W_j\cap\tilde{W}_k} \\
= (\alpha_{i(k)}|_{V_{i(j)}\cap V_{i(k)}})|_{W_j\cap\tilde{W}_k} \qquad &= \alpha_{i(k)}|_{W_j\cap\tilde{W}_k} = (\alpha_{i(k)}|_{\tilde{W}_k})|_{W_j\cap\tilde{W}_k} \\
&= \gamma_k|_{W_j\cap\tilde{W}_k},
\end{aligned}$$

so by definition β and γ define the same differential form.

With the restriction maps the differential forms form a presheaf $A^{p,q}_{X^{\mathrm{an}}}(\bullet)$ on X^{an} by

$$V \mapsto A^{p,q}_{X^{\mathrm{an}}}(V),$$

which is a sheaf by the local nature of Definition 3.4.4.

Lemma 3.4.6. Let $\alpha \in A^{p,q}(V)$ be given by one canonical tropical chart (V, φ_U, α') and assume there exist tropical subcharts $\{(V_i, \varphi_{U_i})\}_{i\in I}$ of (V, φ_U) with $\bigcup_{i\in I} V_i = V$ such that $\alpha|_{V_i} = 0$ for all $i \in I$. Then already $\alpha' = 0$.

Proof. Fix an arbitrary point $x \in \Omega = \mathrm{trop}_U(V)$ and show that α' vanishes in an open neighborhood of x. We choose a relatively compact neighborhood Ω' of x in Ω and let $V' := \mathrm{trop}_U^{-1}(\Omega') \subseteq V$ be its preimage. Since trop_U is proper, V' is relatively compact as well und thus covered by finitely many V_i. Replacing the tropical chart (V, φ_U) by (V', φ_U) and (V_i, φ_{U_i}) by $(V' \cap V_i, \varphi_{U_i})$ and considering the superform $\alpha'|_{\Omega'}$, we may assume that I is finite.

Now by the finiteness of I we are able to choose polyhedral structures $\mathcal{C}$ (respectively $\mathcal{C}_i$) on $\mathrm{Trop}(U)$ (respectively $\mathrm{Trop}(U_i)$) such that for each $\sigma_i \in \mathcal{C}_i$ there exists $\sigma \in \mathcal{C}$ with $\mathrm{Trop}(\psi_{U,U_i})(\sigma_i) = \sigma$. Let $\sigma \in \mathcal{C}$ and we want to show $\alpha'|_{\Omega\cap\sigma} = 0$.

Let $(\sigma_{ij} \in \mathcal{C}_i)_{ij}$ be the finite collection of polyhedra with $\mathrm{Trop}(\psi_{U,U_i})(\sigma_{ij}) = \sigma$, and write $\Omega_i = \mathrm{trop}_{U_i}(V_i)$. Then, as $\mathrm{trop}_U = \mathrm{Trop}(\psi_{U,U_i}) \circ \mathrm{trop}_{U_i}$ on U_i, we have

$\Omega = \bigcup_i \mathrm{Trop}(\psi_{U,U_i})(\Omega_i)$ and so

$$\Omega \cap \sigma = \bigcup_i \mathrm{Trop}(\psi_{U,U_i})(\Omega_i) \cap \sigma = \bigcup_{i,j} \mathrm{Trop}(\psi_{U,U_i})(\Omega_i \cap \sigma_{ij}),$$

where the right hand side is an open cover by Remark 3.3.19 iii.). By the same remark, for all i, j the underlying linear map $\mathbb{L}(\sigma_{ij}) \to \mathbb{L}(\sigma)$ is surjective and as $\mathrm{Trop}(\psi_{U,U_i})^* \alpha'|_{\sigma_{ij} \cap \Omega_i} = 0$ by assumption, we also obtain $\alpha'|_{\mathrm{Trop}(\psi_{U,U_i})(\sigma_{ij} \cap \Omega_i)} = 0$. This shows the claim. □

Remark 3.4.7. The above lemma implies that for any differential form $\alpha \in A^{p,q}(V)$ which is given by a single superform $\alpha_U \in A^{p,q}_{\mathrm{Trop}(U)}(\mathrm{trop}_U(V))$ for a tropical chart (V, φ_U), we have $\alpha = 0$ if and only if $\alpha_U = 0$.

Corollary 3.4.8. Let $V \subseteq X^{\mathrm{an}}$ be an open subset and let $\alpha = \{(V_i, \varphi_{U_i}, \alpha_i)\}_{i \in I}$ and $\alpha' = \{(V'_j, \varphi_{U'_j}, \alpha'_j)\}_{j \in J}$ be two differential forms on V. Suppose there are tropical subcharts $\{(W_{ijl}, \varphi_{\tilde{U}_{ijl}})\}_{ijl}$ of $(V_i \cap V'_j, \varphi_{U_i \cap U'_j})$ for all i, j such that $V_i \cap V'_j = \bigcup_{ijl} W_{ijl}$ with

$$(\alpha_i|_{V_i \cap V'_j})|_{W_{ijl}} = (\alpha'_j|_{V_i \cap V'_j})|_{W_{ijl}}$$

for all l. Then $\alpha = \alpha'$.

Proof. This is a direct consequence of Lemma 3.4.6. □

Remark 3.4.9. Corollary 3.4.8 shows that in Definition 3.4.4 we could have also required that two differential forms are the same if there is *some* common refinement of their covers of tropical charts such that the pullbacks to this refined cover agree.

Definition 3.4.10. Let $\alpha \in A^{p,q}(V)$ for $V \subseteq X^{\mathrm{an}}$ open. Recall that the *support* of the section α in the sheaf theoretic sense is given as

$$\mathrm{supp}(\alpha) := V \setminus \{x \in V \mid \exists \text{ open } V_x \text{ around } x \text{ with } \alpha|_{V_x} = 0\}.$$

We denote by $A^{p,q}_c(V) = A^{p,q}_{X^{\mathrm{an}},c}(V)$ the space of differential forms of bidegree (p, q) with compact support in V.

Lemma 3.4.11. Let (V, φ_U) be a tropical chart of X^{an} with $\Omega = \mathrm{trop}_U(V)$ and suppose $\alpha \in A^{p,q}(V)$ is given by a single superform $\alpha_U \in A^{p,q}_{\mathrm{Trop}(U)}(\Omega)$. Then

$$\mathrm{trop}_U(\mathrm{supp}(\alpha)) = \mathrm{supp}(\alpha_U),$$

where the right hand side is the support of superforms on polyhedral complexes as in Definition 3.2.2 iii.).

Proof. We show both inclusions. Let $x \in \mathrm{supp}(\alpha)$, and we have $V = \mathrm{trop}_U^{-1}(\Omega)$. Choose an arbitrarily small open set $\Omega' \subseteq \Omega$ around $\mathrm{trop}_U(x)$ in $\mathrm{Trop}(U)$ and let $V' := \mathrm{trop}_U^{-1}(\Omega')$. Hence (V', φ_U) is a tropical chart as well. Then $\alpha|_{V'} \neq 0$, which is given by the single superform $\alpha_U|_{\Omega'}$. Remark 3.4.7 implies that $\alpha_U|_{\Omega'}$ is not identically zero, and as Ω' has been chosen arbitrarily small, we obtain $\mathrm{trop}_U(x) \in \mathrm{supp}(\alpha_U)$.
As $\Omega = \mathrm{trop}_U(V)$, for the other inclusion let $\mathrm{trop}_U(x) \in \mathrm{supp}(\alpha_U)$ for some $x \in V$ and we need to show that $x \in \mathrm{supp}(\alpha)$. Let $V' \subseteq V$ be open subset around x. By passing to a smaller neighborhood we can assume that V' comes from a tropical chart $(V', \varphi_{U'})$ with $\Omega' = \mathrm{trop}_{U'}(V')$. Then $\alpha|_{V'}$ is given by the single superform $\mathrm{Trop}(\psi_{U,U'})^* \alpha_U \in A^{p,q}_{\mathrm{Trop}(U')}(\Omega')$. Note that the set

$$\mathrm{Trop}(\psi_{U,U'})(\Omega') = \mathrm{trop}_U(V') \subseteq \mathrm{Trop}(U)$$

is open in $\mathrm{Trop}(U)$ by Remark 3.3.19. As $\alpha_U|_{\mathrm{trop}_U(V')} \neq 0$, we also obtain that $\alpha|_{V'} \neq 0$ which shows the claim. □

We will now examine the support of differential forms in certain cases. For this we need the following involved result.

Proposition 3.4.12. Let $x \in X^{\mathrm{an}}$. Then there exists a very affine open $U \subseteq X$ with $x \in U^{\mathrm{an}}$ such that for any open neighborhood $W \subseteq U^{\mathrm{an}}$ of x, there is a compact neighborhood V of x in W such that $\mathrm{trop}_U(V)$ is a finite union of $d(x)$-dimensional integral Γ-affine polytopes. Here $d(x) = s(x) + t(x)$ as defined in Remark 3.3.2.

Proof. For details see [Gub16, Proposition 4.14]. □

Lemma 3.4.13. Let W be an open subset of X^{an}. Let $\alpha \in A^{p,q}(W)$ and $x \in W$ with $d(x) < \max(p, q)$. Then $x \notin \mathrm{supp}(\alpha)$.

Proof. By Proposition 3.3.20 and shrinking the open neighborhood around x we can as usual assume that W comes from a tropical chart (W, φ_U) and α is given by a single superform $\alpha_U \in A^{p,q}_{\mathrm{Trop}(U)}(\mathrm{trop}_U(W))$. By Proposition 3.4.12 there is a very affine open subset U_x of U with $x \in U_x^{\mathrm{an}}$ and compact neighborhood V_x of x in $U_x^{\mathrm{an}} \cap W$ such that $\mathrm{trop}_{U_x}(V_x)$ is a finite union of $d(x)$-dimensional polytopes. Choose by Proposition 3.3.20 iv.) a tropical chart $(V', \varphi_{U'})$ with $x \in V' \subseteq V_x \subseteq W$ and $U' \subseteq U_x \subseteq U$. Hence we have as usual an affine morphism of tori $\psi_{U,U'}$ with $\varphi_U = \psi_{U,U'} \circ \varphi_{U'}$ on U'. Note that $\alpha|_{V'}$ is given by the superform

$$\mathrm{Trop}(\psi_{U,U'})^* \alpha_U \in A^{p,q}_{\mathrm{Trop}(U')}(\mathrm{trop}_{U'}(V')).$$

We want to show that this superform becomes zero. We observe that by $U' \subseteq U_x \subseteq U$, $V' \subseteq V_x \subseteq W$ we have a factorization

$$\mathrm{Trop}(\psi_{U,U'}) \circ \mathrm{trop}_{U'} = \mathrm{trop}_U = \mathrm{Trop}(\psi_{U,U_x}) \circ \underbrace{\mathrm{Trop}(\psi_{U_x,U'}) \circ \mathrm{trop}_{U'}}_{=\mathrm{trop}_{U_x} \text{ on } U'},$$

on U', hence

$$\mathrm{Trop}(\psi_{U,U'})(\mathrm{trop}_{U'}(V')) = \mathrm{trop}_U(V') = \mathrm{Trop}(\psi_{U,U_x})(\mathrm{trop}_{U_x}(V')).$$

In particular since $V' \subseteq V_x$, we obtain

$$\dim(\mathrm{trop}_U(V')) \leq \dim(\mathrm{trop}_{U_x}(V')) \leq d(x) < \max(p,q).$$

Here by abuse of notation $\dim(\mathrm{trop}_U(V'))$ denotes the maximum dimension of the polyhedra in $\mathrm{Trop}(U)$ intersecting $\mathrm{trop}_U(V')$.

However by $\mathrm{Trop}(\psi_{U,U'})(\mathrm{trop}_{U'}(V')) = \mathrm{trop}_U(V')$ we can conclude by dimensionality that $\mathrm{Trop}(\psi_{U,U'})^*\alpha_U = 0$ on any polyhedron of $\mathrm{Trop}(U')$ intersecting $\mathrm{trop}_{U'}(V')$, and thus $x \notin \mathrm{supp}(\alpha)$. $\square$

Corollary 3.4.14. Let $W \subseteq X^{\mathrm{an}}$ and $\alpha \in A^{p,q}(W)$. If there exists a Zariski open subset $U \subseteq X$ with $\dim(X \setminus U) < \max(p,q)$, then already $\mathrm{supp}(\alpha) \subseteq W \cap U^{\mathrm{an}}$.

Proof. Let $x \in W \setminus U^{\mathrm{an}}$. Then by Remark 3.3.2 we know that $d(x) \leq \dim(X \setminus U) < \max(p,q)$. By Lemma 3.4.13 we get $x \notin \mathrm{supp}(\alpha)$. $\square$

Remark 3.4.15. Let $\alpha \in A^{p,q}(X^{\mathrm{an}})$ with $\max(p,q) = \dim(X)$. Interestingly the support of α is quite small, and can always be found in the valuations on the function field $K(X)$ extending $|\cdot|_K$:

Recall that as X is irreducible, any proper closed subset $Y \subseteq X$ fulfills $\dim Y < \dim X$ (see e.g. [GW10, Lemma 5.7]). Hence by Corollary 3.4.14, the support of α is contained in the analytification of any nonempty open subset of X. If $\pi\colon X^{\mathrm{an}} \to X$ is the usual analytification morphism as in Remark 3.3.1, then we already have

$$\mathrm{supp}(\alpha) \subseteq \pi^{-1}(\{\eta\}),$$

where η is the generic point of X. Indeed, by the above we have

$$\mathrm{supp}(\alpha) \subseteq \bigcap_{\substack{U \subseteq X \text{open} \\ U \neq \emptyset}} \pi^{-1}(U) = \pi^{-1}\Big(\bigcap_{\substack{U \subseteq X \text{open} \\ U \neq \emptyset}} U\Big) = \pi^{-1}(\{\eta\}).$$

Proposition 3.4.16. Let $\alpha \in A_c^{p,q}(X^{\mathrm{an}})$ be a differential form with $\max(p,q) = \dim(X)$. Then there is a very affine open subset U of X such that $\mathrm{supp}(\alpha) \subseteq U^{\mathrm{an}}$ and such that $\alpha|_{U^{\mathrm{an}}}$ is given by a single superform $\alpha_U \in A^{p,q}_{\mathrm{Trop}(U),c}(\mathrm{Trop}(U))$.

Proof. Write $\alpha = \{(V_i, \varphi_{U_i}, \alpha_i)\}_{i\in I}$ with $\bigcup_{i\in I} V_i = X^{\mathrm{an}}$. As $\mathrm{supp}(\alpha) \subseteq X^{\mathrm{an}}$ is compact, α is given by finitely many tropical charts $\{(V_i, \varphi_i)\}_{i=1,\dots,s}$ and $\alpha_i \in A^{p,q}_{\mathrm{Trop}(U_i)}(\Omega_i)$, where $\Omega_i = \mathrm{trop}_{U_i}(V_i)$ is an open subset of $\mathrm{Trop}(U_i)$. The intersection $U = U_1 \cap \dots \cap U_s$ is a nonempty very affine open subset of X. Let V be the open subset of U^{an} given by

$$V := U^{\mathrm{an}} \cap \bigcup_{i=1}^{s} V_i.$$

Furthermore, as X is irreducible, we have $\dim(X \setminus U) < \dim(X)$ for the proper closed subset $X \setminus U$, hence by Corollary 3.4.14 we get $\mathrm{supp}(\alpha) \subseteq U^{\mathrm{an}}$. As $U \subseteq U_i$ for each $i \in \{1,\dots,s\}$ we have as always a refinement

$$\varphi_{U_i} = \psi_{U_i,U} \circ \varphi_U$$

on U with

$$\mathrm{trop}_{U_i} = \mathrm{Trop}(\psi_{U_i,U}) \circ \mathrm{trop}_U$$

on U^{an}. With the above and recalling that $V_i := \mathrm{trop}_{U_i}^{-1}(\Omega_i)$, one checks easily that for all $i \in \{1,\dots,s\}$ we have

$$\mathrm{trop}_U(V_i \cap U^{\mathrm{an}}) = \mathrm{Trop}(\psi_{U_i,U})^{-1}(\Omega_i) \cap \mathrm{Trop}(U)$$

and we denote this open subset of $\mathrm{Trop}(U)$ by Ω_i'. Define $\Omega := \bigcup_{i=1}^{s} \Omega_i' \subseteq \mathrm{Trop}(U)$ and we get $\mathrm{trop}_U^{-1}(\Omega) = V$. This shows that (V, φ_U) is a tropical chart of X^{an}. We consider now the restriction $\alpha|_V \in A^{p,q}(V)$ which is represented by $(\{V_i \cap U^{\mathrm{an}}, \varphi_U, \alpha_i'\})_{i=1,\dots,s}$, where

$$\alpha_i' = \alpha_i|_{V_i \cap U^{\mathrm{an}}} = \mathrm{Trop}(\psi_{U_i,U})^* \alpha_i \in A^{p,q}_{\mathrm{Trop}(U)}(\Omega_i').$$

As $\alpha_i'|_{V_i\cap V_j \cap U^{\mathrm{an}}} = \alpha_j'|_{V_i \cap V_j \cap U^{\mathrm{an}}}$, and as all α_i' are given via the same affine open U, the superforms α_i' glue to a single superform $\alpha_U \in A^{p,q}_{\mathrm{Trop}(U)}(\Omega)$ with $\alpha_U|_{V_i \cap U^{\mathrm{an}}} = \alpha_i'$. By Remark 3.4.9 $\alpha|_V$ is given by the single superform α_U. By Lemma 3.4.11 α_U has compact support in Ω. As $\mathrm{supp}(\alpha) \subseteq U^{\mathrm{an}}$ is already covered by $V_1, \dots, V_s$, we can conclude by extending α_U by zero that α_U is a superform on $\mathrm{Trop}(U)$ which defines $\alpha|_{U^{\mathrm{an}}}$. $\square$

Definition 3.4.17.

i.) Let $\alpha \in A_c^{n,n}(W)$ for an open subset $W \subseteq X^{\mathrm{an}}$ with $n = \dim(X)$. We may view α as an (n,n)-form on X^{an} with compact support. A very affine open subset U as in

Proposition 3.4.16 is called *a very affine chart of integration for* α. Then α is given by a superform $\alpha_U \in A^{n,n}_{\mathrm{Trop}(U),c}(\mathrm{Trop}(U))$.

ii.) We define the *integral of* α *over* W by

$$\int_W \alpha := \int_{\mathrm{Trop}(U)} \alpha_U.$$

Here $\mathrm{Trop}(U)$ is viewed as a tropical cycle (see Definition 3.2.7 and Remark 3.3.6) and we integrate along the polyhedral complex $\mathrm{Trop}(U)$ as in Definition 3.2.5.

Remark 3.4.18. Let $\alpha \in A^{n,n}_c(W)$ for an open subset $W \subseteq X^{\mathrm{an}}$ with $n = \dim(X)$.

i.) Let $U' \subseteq U$ be an inclusion of nonempty very affine subsets of X. If U is a very affine chart of integration for α, then by Corollary 3.4.14 the subset U' is a very affine chart of integration as well.

ii.) The definition of $\int_W \alpha$ is well-defined, i.e. it does not depend on the choice of the very affine chart of integration for α. Indeed, by passing to the intersection of the affine charts of integration, by i.) we can always assume we are in the situation of an inclusion $U' \subseteq U$ as in *i.*). We need to show that

$$\int_{\mathrm{Trop}(U)} \alpha_U = \int_{\mathrm{Trop}(U')} \alpha_{U'},$$

where α is given on U^{an} (respectively $(U')^{\mathrm{an}}$) by $\alpha_U \in A^{n,n}_{\mathrm{Trop}(U),c}(\mathrm{Trop}(U))$ (respectively $\alpha_{U'} \in A^{n,n}_{\mathrm{Trop}(U'),c}(\mathrm{Trop}(U')))$. As always we obtain affine morphism of tori $\psi_{U,U'}$ with $\mathrm{trop}_U = \mathrm{Trop}(\psi_{U,U'}) \circ \mathrm{trop}_{U'}$ on $(U')^{\mathrm{an}}$, and α is given on $(U')^{\mathrm{an}}$ also by $\mathrm{Trop}(\psi_{U,U'})^*\alpha_U$. By Remark 3.4.7 we already have $\alpha_{U'} = \mathrm{Trop}(\psi_{U,U'})^*\alpha_U$.

Then by the *Sturmfels-Tevelev multiplicity formula* the *pushforward* of the weighted complex $\mathrm{Trop}(\psi_{U,U'})_*(\mathrm{Trop}(U'))$ is equal to $\mathrm{Trop}(U)$, and by the *Projection Formula* the integral on pushforwards agrees with the integral of pullbacks, i.e.

$$\int_{\underbrace{\mathrm{Trop}(\psi_{U,U'})_*(\mathrm{Trop}(U'))}_{=\mathrm{Trop}(U)}} \alpha_U = \int_{\mathrm{Trop}(U')} \underbrace{\mathrm{Trop}(\psi_{U,U'})^*\alpha_U}_{=\alpha_{U'}},$$

which shows the claim. For brevity we have not defined pushforwards of weighted complexes, hence have neither stated the Sturmfels-Tevelev multiplicity formula nor the Projection Formula. For definitions and details see [Gub16, 3.9, Proposition 3.10, Proposition 4.11].

iii.) For $\lambda, \rho \in \mathbb{R}$ and $\alpha, \beta \in A_c^{p,q}(W)$ we have

$$\int_W \lambda\alpha + \rho\beta = \lambda \int_W \alpha + \rho \int_W \beta.$$

This follows directly from the corresponding property of integration of superforms along polyhedral complexes after we have chosen a simultaneous very affine chart of integration for both α and β, which is possible by ii.).

Remark 3.4.19. Using the theory on general good K-analytic spaces introduced by Chambert-Loir and Ducros in [CLD12], one can define integration along boundaries of good K-analytic spaces, where we consider boundaries of analytic spaces as introduced in [Ber90, §3.1]. Then there is a version of Stokes' theorem as stated in [CLD12, Théorème 3.12.1]. However the analytification of a variety always has trivial boundary (see [Ber90, Theorem 3.4.1]) and in our setting we obtain the following analogue of Stokes' theorem.

Theorem 3.4.20. Let $n = \dim(X)$, $W \subseteq X^{\mathrm{an}}$ open and $\alpha \in A_c^{n-1,n}(W)$, $\beta \in A_c^{n,n-1}(W)$. We have $\int_W d'\alpha = 0$ and $\int_W d''\beta = 0$.

Proof. By Proposition 3.4.16 there is a very affine subset $U \subseteq X$ such that $\mathrm{supp}(\alpha) \subseteq U^{\mathrm{an}}$ and such that α is given on U^{an} by a single superform $\alpha_U \in A^{p,q}_{\mathrm{Trop}(U),c}(\mathrm{Trop}(U))$. Then U is a very affine chart of integration for $d'\alpha$ via $d'\alpha_U$. The claim follows as the supercurrent $\delta_{\mathrm{Trop}(U)} \in D_{n,n}(\mathbb{R}_U)$ is d'-closed by Remark 3.2.8. □

Remark 3.4.21. On complex manifolds, integration is defined by using a partition of unity with compact supports subordinated to a covering by holomorphic charts. This was not necessary in our setting as the supports of differential forms are 'quite small' and we could define integration by using a single tropical chart. In the following though we will see that there is a notion of a smooth partition of unity for any open covering of a paracompact open subset of X^{an}. In particular we will see that the sheaves $A^{p,q}_{X^{\mathrm{an}}}$ are fine, which is an important result for the study of their sheaf cohomology groups.

Definition 3.4.22. In analogy with differential geometry, set $C^\infty(V) := A^{0,0}(V)$ for an open subset $V \subseteq X^{\mathrm{an}}$ and call an element in $C^\infty(V)$ a *smooth function on V*. This is motivated as every $(0,0)$-form defines a continuous function on V. Indeed, let $f \in A^{0,0}(V)$ be given by a family $\{V_i, \varphi_{U_i}, f_i\}_{i\in I}$, then for $x \in V$ the assignment $f(x) := f_i \circ \mathrm{trop}_{U_i}(x)$ yields a continuous function on V. This is well defined, as $f_i|_{V_i\cap V_j} = f_j|_{V_i\cap V_j}$ by Definition 3.4.4 i.).

Lemma 3.4.23. Let $\underline{\mathbb{R}}$ be the constant sheaf on X^{an} associated to $\mathbb{R}$. Then there is a canonical isomorphism of sheaves

$$\underline{\mathbb{R}} \cong \ker(d''\colon A^{0,0}_{X^{\mathrm{an}}} \to A^{0,1}_{X^{\mathrm{an}}})$$

coming from the sheafification of the morphism which sends a $c \in \mathbb{R}$ to its induced constant function in $C^\infty(V)$ for any open $V \subseteq X^{\mathrm{an}}$.

Proof. For any $f = \{V_i, \varphi_{U_i}, f_i\}_{i \in I} \in C^\infty(V)$ we have $d''f = 0$ if and only if f is locally constant, as the corresponding statement holds in $\mathrm{trop}_{U_i}(V_i) \subseteq \mathbb{R}_{U_i}$ for f_i and any $i \in I$. This immediately yields the well-definedness of the sheaf morphism and its induced isomorphism on stalks. □

Remark 3.4.24.

i.) Recall that a topological space X is called *paracompact* if every open cover has a locally finite open refinement. An open cover $\{U_i\}_{i \in I}$ of X is called *locally finite* if every $x \in X$ has an open neighborhood that intersects only finitely many U_i.

ii.) Every compact space is paracompact. Furthermore every locally compact space X which is σ-compact (i.e. X is the union of countably many compact subspaces) is paracompact. In particular any locally compact and second-countable Hausdorff space is paracompact. Any closed subspace of a paracompact space is paracompact, however not necessarily every open subspace is.

iii.) By ii.) the space X^{an} is paracompact for a variety X, as X^{an} is locally compact and σ-compact. The latter follows as X^{an} is covered by finitely many closed immersions $X^{\mathrm{an}} \to (\mathbb{A}^n)^{\mathrm{an}}$ and as
$$(\mathbb{A}^n)^{\mathrm{an}} = \bigcup_{r \in \mathbb{N}^n} \mathcal{M}(K\{r^{-1}T\})$$
is σ-compact. Here $K\{r^{-1}T\}$ denotes the K-affinoid algebra $K\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\}$ for $r \in \mathbb{N}^n$.

Definition 3.4.25. Let $(W_i)_{i \in I}$ be an open covering of an open subset $W \subseteq X^{\mathrm{an}}$. A smooth *partition of unity* on W with compact supports subordinated to the covering $(W_i)_{i \in I}$ is a family $(\phi_j)_{j \in J}$ of nonnegative smooth functions with compact support on W with the following properties:

i.) The family $(\mathrm{supp}(\phi_j))_{j \in J}$ is locally finite on W.

ii.) We have $\sum_{j \in J} \phi_j \equiv 1$ on W.

iii.) For every $j \in J$, there is $i(j) \in I$ such that $\mathrm{supp}(\phi_j) \subseteq W_{i(j)}$.

Proposition 3.4.26. Let $(W_i)_{i \in I}$ be an open covering of a paracompact open subset W of X^{an}. Then there is a smooth partition of unity $(\phi_j)_{j \in J}$ on W with compact supports subordinated to the covering $(W_i)_{i \in I}$.

Proof. See [Gub16, Proposition 5.10]. □

Definition 3.4.27. We briefly recall the definitions of fine, soft and acyclic sheaves. Let $\mathcal{F}$ be a sheaf of abelian groups on a topological space X.

i.) The sheaf $\mathcal{F}$ is called *soft*, if for any closed set $S \subseteq X$, the restriction map

$$\mathcal{F}(X) \to \varinjlim_{U \supseteq S \text{ open}} \mathcal{F}(U)$$

is surjective.

ii.) The sheaf $\mathcal{F}$ is called *acyclic*, if all higher sheaf cohomology groups vanish, i.e.

$$\mathrm{H}^i(X, \mathcal{F}) = 0$$

for $i > 0$.

iii.) Let X be paracompact and Hausdorff. The sheaf $\mathcal{F}$ is called *fine*, if for any locally finite open cover $\{U_i\}_{i\in I}$ of X, there exists a family of sheaf morphisms $\{\eta_i \colon \mathcal{F} \to \mathcal{F}\}_{i\in I}$, such that

a.) $\sum_{i\in I} \eta_i = \mathrm{id}_{\mathcal{F}}$;

b.) for any $i \in I$ we have $\eta_{i,x} = 0$ for all x in some neighborhood of $X \setminus U_i$. Here $\eta_{i,x} \colon \mathcal{F}_x \to \mathcal{F}_x$ denotes the induced map on stalks.

Remark 3.4.28. By [Wel80, Proposition II.3.5, Theorem II.3.11], a fine sheaf on a paracompact Hausdorff space is soft, and a soft sheaf is acyclic. Recall that acyclic resolutions can be used to compute sheaf cohomology groups (see e.g. [Wel80, Theorem II.3.13]).

Remark 3.4.29. The space X^{an} is paracompact by Remark 3.4.24 and Hausdorff by Remark 3.3.1, hence we are able to talk about fine sheaves on X^{an}.

Corollary 3.4.30. The sheaves $A^{p,q}$ on X^{an} are fine, soft and acyclic.

Proof. Softness and acyclicity are direct consequences of fineness. Fineness follows from Proposition 3.4.26 and as $A^{p,q}$ is a C^∞-module on X^{an} via the multiplication map (for $V \subseteq X^{\mathrm{an}}$ open)

$$C^\infty(V) \times A^{p,q}(V) \to A^{p,q}(V)$$
$$\big(\{V_i, \varphi_{U_i}, f_i\}_{i\in I}, \{V_j, \varphi_{U_j}, \alpha_j\}_{j\in J}\big) \mapsto \{V_i \cap V_j, \varphi_{U_i \cap U_j}, f_i|_{V_i\cap V_j} \cdot \alpha_j|_{V_i \cap V_j}\}_{(i,j)\in I\times J}.$$ □

Remark 3.4.31. Let n be the dimension of the variety X. For any $0 \leq p \leq n$ we have on X^{an} a complex of sheaves

$$0 \to A^{p,0} \xrightarrow{d''} A^{p,1} \xrightarrow{d''} \dots \xrightarrow{d''} A^{p,n} \to 0.$$

Naturally the question arises if this complex is exact. The following theorem affirms this question in positive degrees and was proved by Jell. It can be seen as an analogue of the classical Poincaré Lemma.

Theorem 3.4.32. (d''-Poincaré Lemma) Let $V \subseteq X^{\mathrm{an}}$ be an open subset. Let $x \in V$ and $\alpha \in A^{p,q}(V)$ with $q > 0$ and $d''\alpha = 0$. Then there exists some open $W \subseteq V$ with $x \in W$ and some $\beta \in A^{p,q-1}(W)$ such that $d''\beta = \alpha|_W$.

Proof. See [Jel16a, Theorem 4.5]. □

Corollary 3.4.33. (Dolbeault Cohomology) Let X be a variety of dimension n. For any $0 \leq p \leq n$ the complex

$$0 \to A^{p,0} \xrightarrow{d''} A^{p,1} \xrightarrow{d''} \dots \xrightarrow{d''} A^{p,n} \to 0$$

is exact in positive degrees by Theorem 3.4.32 and yields an acyclic resolution of the kernel sheaf

$$0 \to \ker(d''\colon A^{p,0}_{X^{\mathrm{an}}} \to A^{p,1}_{X^{\mathrm{an}}}) \to A^{p,\bullet}_{X^{\mathrm{an}}},$$

and hence the sheaves $A^{p,\bullet}_{X^{\mathrm{an}}}$ can be used to compute the sheaf cohomology groups of $\ker(d''\colon A^{p,0}_{X^{\mathrm{an}}} \to A^{p,1}_{X^{\mathrm{an}}})$.

We define the *Dolbeault cohomology groups* $\mathrm{H}^{p,q}(X^{\mathrm{an}})$ of X^{an} as

$$\mathrm{H}^{p,q}(X^{\mathrm{an}}) := \frac{\ker(d''\colon A^{p,q}(X^{\mathrm{an}}) \to A^{p,q+1}(X^{\mathrm{an}}))}{\mathrm{im}(d''\colon A^{p,q-1}(X^{\mathrm{an}}) \to A^{p,q}(X^{\mathrm{an}}))}. \tag{3.4.1}$$

Here for consistency of notation we set $A^{p,q}_{X^{\mathrm{an}}} = 0$ for $q < 0$. With the above we obtain isomorphisms

$$\mathrm{H}^{\bullet}(X^{\mathrm{an}}, \ker(d''\colon A^{p,0}_{X^{\mathrm{an}}} \to A^{p,1}_{X^{\mathrm{an}}})) \cong \mathrm{H}^{p,\bullet}(X^{\mathrm{an}}),$$

and in particular with Lemma 3.4.23 isomorphisms

$$\mathrm{H}^{\bullet}(X^{\mathrm{an}}, \underline{\mathbb{R}}) \cong \mathrm{H}^{0,\bullet}(X^{\mathrm{an}}).$$

4

The Tropical Cycle Class Map

In this final chapter we draw the connection between Chapter 2 and Chapter 3. In particular we will construct a morphism from the sheaf of rational Milnor K-Theory of $\mathcal{O}_{X^{\mathrm{an}}}$ to the kernel sheaf of differential forms on X^{an}. This connection will then be used to finally define the tropical cycle class map $\mathrm{cl}_{\mathrm{trop}} \colon \mathrm{CH}^p(X) \to \mathrm{H}^{p,p}(X^{\mathrm{an}})$ and formulate Theorem 4.2.5 by Liu.

4.1 The map from Milnor K-Theory to differential forms

The aim of this section is to define a $\mathbb{Q}$-linear morphism of sheaves

$$\mathscr{K}^p_{X^{\mathrm{an}}} \to \ker(d'' \colon A^{p,0}_{X^{\mathrm{an}}} \to A^{p,1}_{X^{\mathrm{an}}}).$$

Yet before this, we need to make some technical remarks:

In [CLD12], differential forms and moment maps are defined for general good K-analytic spaces. In the following we will see that for the analytification of a variety, the tropicalization of a moment map in the sense of [CLD12] locally fits with our algebraic approach and we use this to establish a morphism between the sheaf of rational Milnor K-Theory $\mathscr{K}^p_{X^{\mathrm{an}}}$ and the kernel sheaf $\ker(d'' \colon A^{p,0}_{X^{\mathrm{an}}} \to A^{p,1}_{X^{\mathrm{an}}})$. We assume all K-analytic spaces in the sense of Berkovich to be good.

Definition 4.1.1. (Moment maps on analytic spaces) Let X be a K-analytic space. We call a morphism $\varphi \colon X \to (\mathbb{G}_m^q)^{\mathrm{an}}$ of K-analytic spaces an *analytic moment map*. As in the case of algebraic varieties, call $\varphi_{\mathrm{trop}} := \mathrm{trop} \circ \varphi \colon X \to \mathbb{R}^q$ the *tropicalization map* of φ.

Remark 4.1.2. Let $(X, \mathcal{O}_X)$ be a K-analytic space and $U \subseteq X$ open. We can fully classify analytic moment maps $U \to (\mathbb{G}_m^q)^{\mathrm{an}}$ by invertible analytic functions. Indeed, let

$f_1, \dots, f_q \in \mathcal{O}_X(U)^*$. These elements induce an analytic moment map

$$G\colon U \to (\mathbb{G}_m^q)^{\mathrm{an}}$$

as follows:

As $\mathbb{G}_m^q = \operatorname{Spec}(K[T_1^{\pm 1}, \dots, T_q^{\pm 1}])$ is an affine scheme, we have the standard bijection $\operatorname{Hom}_{\mathrm{LRS}/K}(X, \mathbb{G}_m^q) \cong \operatorname{Hom}_{K\text{-alg.}}(K[T_1^{\pm 1}, \dots, T_q^{\pm 1}], \mathcal{O}_X(U))$. Hence, as the invertible analytic functions f_i yield a K-algebra morphism

$$\begin{aligned} \varphi\colon K[T_1^{\pm 1}, \dots, T_q^{\pm 1}] &\to \mathcal{O}_X(U), \\ T_i &\mapsto f_i \end{aligned}$$

we get an induced morphism of locally K-ringed spaces $F\colon U \to \mathbb{G}_m^q$. Note that the image $F(p)$ of a $p \in U$ is given as the preimage of the maximal ideal $\mathfrak{m}_{X,p} \subseteq \mathcal{O}_{X,p}$ under the map

$$K[T_1^{\pm 1}, \dots, T_q^{\pm 1}] \xrightarrow{\varphi} \mathcal{O}_X(U) \longrightarrow \mathcal{O}_{X,p}. \tag{4.1.1}$$

As known by Remark 3.3.1, there exists a unique morphism $G\colon U \to (\mathbb{G}_m^q)^{\mathrm{an}}$ of K-analytic spaces, such that the following diagram commutes

$$\begin{tikzcd} & (\mathbb{G}_m^q)^{\mathrm{an}} \arrow[dr, "\pi"] & \\ U \arrow[ur, "G"] \arrow[rr, "F"] & & \mathbb{G}_m^q. \end{tikzcd}$$

Here π is the usual analytification map $p \mapsto \ker(p)$ for a multiplicative seminorm $p \in (\mathbb{G}_m^q)^{\mathrm{an}}$ as in Remark 3.3.1. Using the diagram above, equation (4.1.1) and by uniqueness, we can give an explicit description of the morphism G on topological spaces as follows:

Let $p \in U$. By passing to a smaller open subset in the K-analytic atlas of X, we can assume that U is an open subset of $\mathcal{M}(A)$, where A is a K-affinoid algebra. Then for $p \in U \subseteq \mathcal{M}(A)$ we get that $G(p) \in (\mathbb{G}_m^q)^{\mathrm{an}}$ is the multiplicative seminorm which maps a $g \in K[T_1^{\pm 1}, \dots, T_q^{\pm 1}]$ to

$$|g(G(p))| = |\varphi(g)_p(p)|. \tag{4.1.2}$$

Here the right hand side is the absolute value of the image of the germ $\varphi(g)_p \in \mathcal{O}_{X,p}$ under the usual map $\mathcal{O}_{X,p} \to \mathcal{H}(p)$, where $\mathcal{H}(p)$ is the completion of $\operatorname{Frac}(A/\ker(p))$ with respect to p. Note that $\mathfrak{m}_{X,p} = \{f \in \mathcal{O}_{X,p} \mid |f_p(p)| = 0\}$.

On the other hand, any analytic morphism $G\colon U \to (\mathbb{G}_m^q)^{\mathrm{an}}$ gives a morphism of locally ringed spaces $F = \pi \circ G$ as in the diagram above and hence we obtain elements $F^{\#}(U)(T_i) \in \mathcal{O}_X(U)^*$ for $i = 1, \dots, q$. Uniqueness in the discussion above shows that G must already come from these elements.

The following result states that the our algebraic moment maps locally fit with the analytic moment maps after tropicalization.

Proposition 4.1.3. Let X be an algebraic variety over K and $T = \mathbb{G}_m^q$ a split torus. Let $\varphi\colon W \to T^{\mathrm{an}}$ be an analytic moment map defined on an open subset W of X^{an}. For every $x \in W$ there is a very affine open subset U of X with an algebraic moment map $\varphi'\colon U \to T$ and an open neighborhood V of x in $U^{\mathrm{an}} \cap W$ such that $\varphi_{\mathrm{trop}} = \varphi'_{\mathrm{trop}}$ on V. Recall that $\varphi'_{\mathrm{trop}} = \mathrm{trop} \circ (\varphi')^{\mathrm{an}}$.

Proof. As in Remark 4.1.2 the analytic moment map φ is given by invertible analytic functions $\varphi_1, \dots, \varphi_q$. The general idea of the proof is then to see that the $\varphi_i' s$ restrict to strictly convergent Laurent series in some suitable Laurent domain V in W. After cutting these series off in sufficiently high positive and negative degree, we obtain Laurent polynomials whose valuations match with the φ_i in V, and can thus be used to define the desired algebraic moment map. For details see [Gub16, Proposition 7.2]. □

The next lemma is slightly technical, however is important for subsequent constructions.

Lemma 4.1.4. Let $W \subseteq X^{\mathrm{an}}$ open and $\Phi\colon W \to (\mathbb{G}_m^n)^{\mathrm{an}}$ an analytic moment map. Furthermore let α be a (p,q)-superform on $\mathbb{R}^n$. Then there is an $\omega \in A^{p,q}_{X^{\mathrm{an}}}(W)$ which is locally given by α, i.e. for all $x \in W$ there exists a tropical chart (V, φ_U) with $x \in V \subseteq W$ and integral Γ-affine map $F\colon \mathbb{R}_U \to \mathbb{R}^n$ such that

$$\Phi_{\mathrm{trop}} = F \circ \mathrm{trop}_U$$

on V and

$$\omega|_V = F^*\alpha \in A^{p,q}_{\mathrm{Trop}(U)}(\mathrm{trop}_U(V)).$$

Proof. Let $x \in W$ and we choose a triple $\tilde{V}, \tilde{U}, \tilde{\varphi}$ as in Proposition 4.1.3 with $\tilde{\varphi}_{\mathrm{trop}} = \Phi_{\mathrm{trop}}$ on $\tilde{V}$. By Proposition 3.3.20 iv.) choose tropical chart (V_x, φ_{U_x}) with $V_x \subseteq \tilde{V}$ and $U_x \subseteq \tilde{U}$. Then φ_{U_x} refines $\tilde{\varphi}$, i.e. there exists an affine morphism of tori $\psi_x\colon T_{U_x} \to \mathbb{G}_m^n$ with $\tilde{\varphi} = \psi_x \circ \varphi_{U_x}$ on U_x. Write F_x for the Γ-affine map $\mathrm{Trop}(\psi_x)\colon \mathbb{R}_{U_x} \to \mathbb{R}^n$ and in particular we obtain (as $V_x \subseteq \tilde{V}$, and $\Phi_{\mathrm{trop}} = \tilde{\varphi}_{\mathrm{trop}}$ on $\tilde{V}$) that

$$\Phi_{\mathrm{trop}} = F_x \circ \mathrm{trop}_{U_x} \tag{4.1.3}$$

on V_x. We write as usual $\Omega_x = \operatorname{trop}_{U_x}(V_x)$ for the associated open subset of the polyhedral complex $\operatorname{Trop}(U_x)$. Define

$$\omega_x := F_x^* \alpha \in A^{p,q}_{\operatorname{Trop}(U_x)}(\Omega_x) \tag{4.1.4}$$

and together a differential form

$$\omega := \{(V_x, \varphi_{U_x}, \omega_x)\}_{x \in W} \in A^{p,q}(W).$$

It remains to show that ω lies indeed in $A^{p,q}(W)$, i.e. we need to show that for $x, y \in W$ we have

$$\omega_x|_{V_x \cap V_y} = \omega_y|_{V_x \cap V_y}.$$

As above write $\Omega_{x,y} = \operatorname{trop}_{U_x \cap U_y}(V_x \cap V_y)$. Let $z \in \Omega_{x,y}$ and σ be a polyhedron in $\operatorname{Trop}(U_x \cap U_y)$ which contains z. Furthermore let $v_1, \dots, v_p, w_1, \dots, w_q \in \mathbb{L}_\sigma$. Note that we can assume for any $w \in \mathbb{L}_\sigma$ after sufficiently scaling down (pass to $c \cdot w$ for a small $c > 0$) that there is $a_w, b_w \in \Omega_{x,y}$ such that $w = a_w - b_w$. To ease notation write G_x (and analogously G_y) for the affine map

$$F_x \circ \operatorname{Trop}(\psi_{U_x, U_x \cap U_y}) \colon \mathbb{R}_{U_x \cap U_y} \to \mathbb{R}^n$$

with linear part g_x. Note that we have with (4.1.3) that

$$G_x = G_y$$

on $\Omega_{x,y}$, hence $G_x(z) = G_y(z)$ and for $w \in \mathbb{L}_\sigma$ as above

$$g_x(w) = g_x(a_w - b_w) \overset{G_x \text{ affine}}{=} G_x(a_w) - G_x(b_w) \overset{a_w, b_w \in \Omega_{x,y}}{=} G_y(a_w) - G_y(b_w) = g_y(w).$$

This yields immediately that

$$\begin{aligned}
\langle \omega_x|_{V_x \cap V_y}(z); v_1, \dots, v_p, w_1, \dots, w_q \rangle &= \\
\langle \operatorname{Trop}(\psi_{U_x, U_x \cap U_y})^* F_x^* \alpha(z); v_1, \dots, v_p, w_1, \dots, w_q \rangle &= \\
\langle \alpha(G_x(z)); g_x(v_1), \dots, g_x(v_p), g_x(w_1), \dots, g_x(w_q) \rangle &= \\
\langle \alpha(G_y(z)); g_y(v_1), \dots, g_y(v_p), g_y(w_1), \dots, g_y(w_q) \rangle &= \\
\langle \omega_y|_{V_x \cap V_y}(z); v_1, \dots, v_p, w_1, \dots, w_q \rangle&,
\end{aligned}$$

which shows the claim. The above discussion also shows that ω does not depend on the choice of the tropical charts around $x \in W$. $\square$

Definition 4.1.5. Let X be an algebraic variety over K with corresponding analytification $(X^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}})$. Furthermore let $\mathscr{K}^p_{X^{\mathrm{an}}}$ be the p-th sheaf of rational Milnor K-Theory of $\mathcal{O}_{X^{\mathrm{an}}}$

as introduced in Definition 2.3.2. We define a $\mathbb{Q}$-linear map of sheaves

$$\tau^p_{X^{\mathrm{an}}} \colon \mathscr{K}^p_{X^{\mathrm{an}}} \to \ker(d'' \colon A^{p,0}_{X^{\mathrm{an}}} \to A^{p,1}_{X^{\mathrm{an}}})$$

as follows. We construct a morphism of presheaves $K^p_M(\mathcal{O}_{X^{\mathrm{an}}}(\bullet)) \otimes \mathbb{Q} \to \ker(d'' \colon A^{p,0}(\bullet) \to A^{p,1}(\bullet))$, the claim follows then from the universal property of the sheafification.

Let $W \subseteq X^{\mathrm{an}}$ be open and $\{f_1, \ldots, f_p\} \in K^M_q(\mathcal{O}_X(W))$ with $f_1, \ldots, f_p \in \mathcal{O}_{X^{\mathrm{an}}}(W)^*$. As in Remark 4.1.2, the f_i's induce a morphism $\Phi \colon W \to (\mathbb{G}^p_m)^{\mathrm{an}}$ of K-analytic spaces. Note that we have

$$\Phi_{\mathrm{trop}}(q) = (-\log |(f_i)_q(q)|)_{i=1,\ldots,p}$$

as in (4.1.2) for $q \in W$.

Endow $\mathbb{R}^p$ with coordinates x_i. The superform $d'x_1 \wedge \cdots \wedge d'x_p \in A^{p,0}(\mathbb{R}^p)$ induces by Lemma 4.1.4 a differential form in $\ker(d'' \colon A^{p,0}(W) \to A^{p,1}(W))$ which we denote by $\tau^p_{X^{\mathrm{an}}}(\{f_1, \ldots, f_p\})$.

Obviously $\tau^p_{X^{\mathrm{an}}}$ is compatible with restriction.

It remains to show that $\tau^p_{X^{\mathrm{an}}}$ factors through the relations of Milnor K-theories.

i.) The fact that

$$\tau^p_{X^{\mathrm{an}}}(\{f_1, \ldots, f_i f_i', \ldots, f_p\}) = \tau^p_{X^{\mathrm{an}}}(\{f_1, \ldots, f_i, \ldots, f_p\}) + \tau^p_{X^{\mathrm{an}}}(\{f_1, \ldots, f_i', \ldots, f_p\})$$

follows from a similar argument as in Lemma 4.1.4 above (i.e. being able to restrict attention to elements in Ω_x after scaling) and noting that for vectors

$$a_j = (-\log |(f_i)_{q_j}(q_j)|)_{i=1,\ldots,p} \in \mathbb{R}^p$$

for $j = 1, \ldots, p$ and $q_j \in W$ we have

$$\begin{aligned} &d'x_1 \wedge \cdots \wedge d'x_p(a_1, \ldots, a_p) = \\ &d'x_1 \wedge \cdots \wedge d'x_p\Big((-\log |(f_1)_{q_j}(q_j)|, \ldots, -\log |(f_i f_i')_{q_j}(q_j)|, \ldots, -\log |(f_p)_{q_j}(q_j)|)_{j=1,\ldots,p}\Big) = \\ &\det\Big((-\log |(f_1)_{q_j}(q_j)|, \ldots, \underbrace{-\log |(f_i f_i')_{q_j}(q_j)|}_{=-\log |(f_i)_{q_j}(q_j)| - \log |(f_i')_{q_j}(q_j)|}, \ldots, -\log |(f_p)_{q_j}(q_j)|)_{j=1,\ldots,p}\Big). \end{aligned}$$

The claim then follows as the determinant is multilinear in rows.

ii.) We need to show that for $j < k \in \{1, \ldots, p\}$ we have

$$\tau^p_{X^{\mathrm{an}}}(\{f_1, \ldots, \underbrace{f}_{j\text{-th}}, \ldots, \underbrace{1-f}_{k\text{-th}}, \ldots, f_p\}) = 0.$$

For $f \in \mathcal{O}_X(W)^*$ and $q \in W$ it follows from the ultrametric triangle inequality and principle of dominance, that for the pair $(f, 1-f)$ we have

$$(-\log|f_q(q)|, -\log|(1-f)_q(q)|) = \begin{cases} (0, -\log|(1-f)_q(q)|) & \text{for } |f_q(q)| = 1, \\ (-\log|f_q(q)|, -\log|f_q(q)|) & \text{for } |f_q(q)| > 1, \\ (-\log|f_q(q)|, 0) & \text{for } |f_q(q)| < 1. \end{cases}$$

Hence $\Phi_{\mathrm{trop}}(W)$ must be contained in the union of the three hypersurfaces

$$\{(x_1, \ldots, x_{j-1}, 0, x_{j+1}, \ldots, x_k, \ldots, x_p) \mid x_i \in \mathbb{R}\} \cup$$
$$\{(x_1, \ldots, x_j, \ldots, x_{k-1}, x_j, x_{k+1} \ldots, x_p) \mid x_i \in \mathbb{R}\} \cup$$
$$\{(x_1, \ldots, x_j, \ldots, x_{k-1}, 0, x_{k+1}, \ldots, x_p) \mid x_i \in \mathbb{R}\}$$

in $\mathbb{R}^p$. Thus any polyhedron contained in $\Phi_{\mathrm{trop}}(W)$ can have dimension at most $p-1$. As $\tau^p_{X^{\mathrm{an}}}$ is by construction locally defined via pullbacks (see equations (4.1.3) and (4.1.4)), this shows that

$$\tau^p_{X^{\mathrm{an}}}(\{f_1, \ldots, f, \ldots, 1-f \ldots, f_p\}) = 0.$$

Finally we set

$$\mathscr{T}^p_{X^{\mathrm{an}}} = \mathscr{K}^p_{X^{\mathrm{an}}} / \ker \tau^p_{X^{\mathrm{an}}},$$

which we consider as a rational subsheaf of $\ker(d''\colon A^{p,0}_{X^{\mathrm{an}}} \to A^{p,1}_{X^{\mathrm{an}}})$.

For the next proposition, we recall two basic facts about base change and tensor products, and define *basic open* subsets of polyhedral complexes as in [JSS19, Definition 3.7].

Remark 4.1.6. Let L/K be a field extension and W a finite K-vector space. We recall:

i.) We have

$$\mathrm{Hom}_K(W, K) \otimes_K L \cong \mathrm{Hom}_K(W, L) \tag{4.1.5}$$

as K-vector spaces. This statement is in general false if both vector spaces L and W are infinite dimensional, however note here that it is enough for only W to be of finite dimension (L/K does not have to be finite).

ii.) There is a canonical group isomorphism

$$\mathrm{Hom}_K(W, L) \cong \mathrm{Hom}_L(W \otimes_K L, L). \tag{4.1.6}$$

This fact follows from a more general statement on extension of scalars of modules.

Definition 4.1.7.

i.) Recall that a subset $\Delta \subseteq \mathbb{R}^r$ is an *open cube* if it is a product of intervals which are of the form (a_i, b_i) for $a_i, b_i \in \mathbb{R}$.

ii.) Let $\mathscr{C}$ be a polyhedral complex in $\mathbb{R}^r$. An open subset Ω of $|\mathscr{C}|$ is called a *basic open* subset if there exists an open cube $\Delta \subseteq \mathbb{R}^r$ such that $\Omega = \Delta \cap |\mathscr{C}|$ and such that the set of polyhedra of $\mathscr{C}$ intersecting Ω has a unique minimal element.

Proposition 4.1.8. Let X be a variety over K and X^{an} its analytification. Then τ_X^p induces an isomorphism of sheaves

$$\mathscr{T}^p_{X^{\mathrm{an}}} \otimes_{\mathbb{Q}} \mathbb{R} \cong \ker(d'' \colon A^{p,0}_{X^{\mathrm{an}}} \to A^{p,1}_{X^{\mathrm{an}}}).$$

Proof. It suffices to show that the induced map on stalks

$$\tau_x \colon \mathscr{T}^p_{X^{\mathrm{an}},x} \otimes_{\mathbb{Q}} \mathbb{R} \to \ker(d'' \colon A^{p,0}_{X^{\mathrm{an}},x} \to A^{p,1}_{X^{\mathrm{an}},x})$$

is an isomorphism for every $x \in X^{\mathrm{an}}$.

For this let $x \in X^{\mathrm{an}}$ and $V \subseteq X^{\mathrm{an}}$ open around x. As the tropical charts form a basis on X^{an}, we can assume that V comes from a tropical chart $(V, \varphi_U \colon U \to \mathbb{G}_m^N)$ for a very affine open $U \subseteq X$ with $\Omega := \mathrm{trop}_U(V)$ open in the polyhedral complex $\mathrm{Trop}(U) \subseteq \mathbb{R}^N$. Denote by $f_1, \ldots, f_N \subseteq \mathcal{O}_X(U)^*$ the elements defining the canonical moment map, i.e. φ_U comes from the K-algebra morphism

$$K[T_1^{\pm 1}, \ldots, T_N^{\pm 1}] \to \mathcal{O}_X(U),\ T_i \mapsto f_i.$$

Additionally keep in mind that we have an embedding $\mathcal{O}_X(U) \hookrightarrow \mathcal{O}_{X^{\mathrm{an}}}(U^{\mathrm{an}})$ for any open $U \subseteq X$ induced by $\pi \colon X^{\mathrm{an}} \to X$ (see Remark 3.3.1).

By [JSS19, Lemma 3.8] the basic open sets form a basis of the topology on the support $|\mathscr{C}|$ of $\mathscr{C} := \mathrm{Trop}(U)$, hence after shrinking V we can assume that $\Omega \subseteq |\mathscr{C}|$ is a basic open set with minimal polyhedron σ. Such a tropical chart (V, φ_U) will be called a *basic tropical chart*. For any $\rho \in \mathscr{C}$ with $\sigma \prec \rho$ we denote as usual (see Definition 3.1.13) by $\mathbb{L}_\rho = \mathbb{Z}_\rho \otimes_{\mathbb{Z}} \mathbb{R} \subseteq \mathbb{R}^N$

the tangent space of ρ, and by $\mathbb{L}_{\rho,\mathbb{Q}} := \mathbb{Z}_\rho \otimes_{\mathbb{Z}} \mathbb{Q}$ its underlying $\mathbb{Q}$-linear subspace in $\mathbb{Q}^N$. Here $\mathbb{Z}_\rho$ is as usual the canonical lattice of ρ. Note that $\mathbb{L}_{\rho,\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{L}_\rho$ as $\mathbb{R}$-vector spaces.

We have an inclusion of finite $\mathbb{Q}$-vector spaces

$$\sum_{\rho \in \mathscr{C}, \sigma \prec \rho} \bigwedge^p \mathbb{L}_{\rho,\mathbb{Q}} \subseteq \bigwedge^p \mathbb{Q}^N,$$

and thus a surjective map

$$r\colon \mathrm{Hom}_{\mathbb{Q}}\left(\bigwedge^p \mathbb{Q}^N, \mathbb{R}\right) \to \mathrm{Hom}_{\mathbb{Q}}\left(\sum_{\rho \in \mathscr{C}, \sigma \prec \rho} \bigwedge^p \mathbb{L}_{\rho,\mathbb{Q}}, \mathbb{R}\right). \tag{4.1.7}$$

By [JSS19, Proposition 3.20] we have an isomorphism of $\mathbb{R}$-vector spaces

$$\mathrm{Hom}_{\mathbb{R}}\left(\sum_{\rho \in \mathscr{C}, \sigma \prec \rho} \bigwedge^p \mathbb{L}_{\rho}, \mathbb{R}\right) \xrightarrow{\sim} \ker(d''\colon A^{p,0}_{\mathscr{C}}(\Omega) \to A^{p,1}_{\mathscr{C}}(\Omega)). \tag{4.1.8}$$

As base change commutes with taking exterior products, we obtain by Remark 4.1.6 ii.) that

$$\begin{aligned} \mathrm{Hom}_{\mathbb{R}}\left(\sum_{\rho \in \mathscr{C}, \sigma \prec \rho} \bigwedge^p \mathbb{L}_{\rho}, \mathbb{R}\right) &\cong \mathrm{Hom}_{\mathbb{R}}\left(\left(\sum_{\rho \in \mathscr{C}, \sigma \prec \rho} \bigwedge^p \mathbb{L}_{\rho,\mathbb{Q}}\right) \otimes_{\mathbb{Q}} \mathbb{R}, \mathbb{R}\right) \\ &\overset{(4.1.6)}{\cong} \mathrm{Hom}_{\mathbb{Q}}\left(\sum_{\rho \in \mathscr{C}, \sigma \prec \rho} \bigwedge^p \mathbb{L}_{\rho,\mathbb{Q}}, \mathbb{R}\right). \end{aligned}$$

Thus the canonical map

$$\mathrm{Hom}_{\mathbb{Q}}\left(\bigwedge^p \mathbb{Q}^N, \mathbb{R}\right) \to \ker(d''\colon A^{p,0}_{\mathscr{C}}(\Omega) \to A^{p,1}_{\mathscr{C}}(\Omega))$$

factors through r in (4.1.7) and also induces an isomorphism of $\mathbb{Q}$-vector spaces

$$\mathrm{Hom}_{\mathbb{Q}}\left(\sum_{\rho \in \mathscr{C}, \sigma \prec \rho} \bigwedge^p L_{\rho,\mathbb{Q}}, \mathbb{R}\right) \xrightarrow{\sim} \ker(d''\colon A^{p,0}_{\mathscr{C}}(\Omega) \to A^{p,1}_{\mathscr{C}}(\Omega)).$$

We consider now the $\mathbb{Q}$-linear subspace $\mathscr{T}^p_{X^{\mathrm{an}}}(V)_{f_1,\dots,f_N}$ of $\mathscr{T}^p_{X^{\mathrm{an}}}(V)$ which is generated by the elements in $\mathscr{K}^p_{X^{\mathrm{an}}}(V)$ given by $\{f_{i_1}, \dots, f_{i_p}\} \in K^p_M(\mathcal{O}_{X^{\mathrm{an}}}(V))$, where $\{i_1 < \cdots < i_p\}$ is an ordered subset of $\{1, \dots, N\}$. Here we denote ambiguously by abuse of notation $f_j \in \mathcal{O}_{X^{\mathrm{an}}}(V)$ as the restrictions of the $f_j \in \mathcal{O}_X(U) \hookrightarrow \mathcal{O}_{X^{\mathrm{an}}}(U^{\mathrm{an}})$ to $V \subseteq U^{\mathrm{an}}$.

Let $J = \{i_1, \dots, i_p\} \subseteq \{1, \dots, N\}$ be such an ordered subset. We now want to describe the image $\tau^p_{X^{\mathrm{an}}}(\{f_{i_1},\dots,f_{i_p}\}) \in \ker(d''\colon A^{p,0}_{X^{\mathrm{an}}}(V) \to A^{p,1}_{X^{\mathrm{an}}}(V))$. The elements $f_{i_1},\dots,f_{i_p}$ induce the tropical morphism

$$\mathrm{trop}_{U,J}\colon V \to \mathbb{R}^p,\ |\cdot| \mapsto (-\log|f_{i_1}|,\dots,-\log|f_{i_p}|).$$

Note that we have $\mathrm{trop}_{U,J} = \mathrm{proj}_J \circ \mathrm{trop}_U$ on V, where $\mathrm{proj}_J\colon \mathbb{R}^N \to \mathbb{R}^p$ is simply the projection onto the coordinates given by J. Thus by construction as in Lemma 4.1.4, the differential form $\tau^p_{X^{\mathrm{an}}}(\{f_{i_1},\dots,f_{i_p}\}) \in \ker(d''\colon A^{p,0}_{X^{\mathrm{an}}}(V) \to A^{p,1}_{X^{\mathrm{an}}}(V))$ is given by the single superform

$$\mathrm{proj}_J^*(d'x_1 \wedge \cdots \wedge d'x_p) = d'x_{i_1} \wedge \cdots \wedge d'x_{i_p} \in \ker(d''\colon A^{p,0}_{\mathscr{C}}(\Omega) \to A^{p,1}_{\mathscr{C}}(\Omega)).$$

Recall that $\mathscr{T}^p_{X^{\mathrm{an}}} = \mathscr{K}^p_{X^{\mathrm{an}}}/\ker(\tau^p_{X^{\mathrm{an}}})$. As in (4.1.7) the injective map of $\mathbb{Q}$-vector spaces

$$\tau^p_{X^{\mathrm{an}}}\colon \mathscr{T}^p_{X^{\mathrm{an}}}(V)_{f_1,\dots,f_N} \hookrightarrow \ker(d''\colon A^{p,0}_{\mathscr{C}}(\Omega) \to A^{p,1}_{\mathscr{C}}(\Omega)) \tag{4.1.9}$$

factors through $\mathrm{Hom}_{\mathbb{Q}}\left(\sum_{\rho\in\mathscr{C},\sigma\prec\rho} \bigwedge^p \mathbb{L}_{\rho,\mathbb{Q}}, \mathbb{R}\right) \cong \ker(d''\colon A^{p,0}_{\mathscr{C}}(\Omega) \to A^{p,1}_{\mathscr{C}}(\Omega))$. The $d'x_{i_1} \wedge \cdots \wedge d'x_{i_p}$ already form a $\mathbb{Q}$-basis for $\bigwedge^p \mathbb{Q}^N$, thus (4.1.9) induces an isomorphism of $\mathbb{Q}$-vector spaces

$$\mathscr{T}^p_{X^{\mathrm{an}}}(V)_{f_1,\dots,f_N} \cong \mathrm{Hom}_{\mathbb{Q}}\left(\sum_{\rho\in\mathscr{C},\sigma\prec\rho} \bigwedge^p \mathbb{L}_{\rho,\mathbb{Q}}, \mathbb{Q}\right). \tag{4.1.10}$$

Observe that we only get an isomorphism to the space of linear morphisms to $\mathbb{Q}$ on the right hand side. Taking the tensor product with $\mathbb{R}$ on both sides finally yields together with Remark 4.1.6 and (4.1.8) an isomorphism of $\mathbb{R}$-vector spaces

$$\mathscr{T}^p_{X^{\mathrm{an}}}(V)_{f_1,\dots,f_N} \otimes_{\mathbb{Q}} \mathbb{R} \cong \ker(d''\colon A^{p,0}_{\mathscr{C}}(\Omega) \to A^{p,1}_{\mathscr{C}}(\Omega)).$$

By taking the colimit over all basic tropical charts at x, we can conclude the isomorphism τ_x on stalks. $\square$

Corollary 4.1.9. Let X be a variety over K and X^{an} its analytification. For all $p, q \geq 0$, we have a canonical isomorphism

$$\mathrm{H}^q(X^{\mathrm{an}}, \mathscr{T}^p_{X^{\mathrm{an}}}) \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathrm{H}^{p,q}(X^{\mathrm{an}}),$$

where the right hand side denotes the Dolbeault cohomology groups as in (3.4.1). In particular this endows the real vector space $\mathrm{H}^{p,q}(X^{\mathrm{an}})$ with a canonical rational structure.

Proof. By Proposition 4.1.8 and Corollary 3.4.33 we have

$$\begin{aligned}\mathrm{H}^{p,q}(X^{\mathrm{an}}) \cong \mathrm{H}^q(X^{\mathrm{an}}, \ker(d''\colon A^{p,0} \to A^{p,1})) &\cong \mathrm{H}^q(X^{\mathrm{an}}, \mathscr{T}^p_{X^{\mathrm{an}}} \otimes_{\mathbb{Q}} \mathbb{R}) \\ &\cong \mathrm{H}^q(X^{\mathrm{an}}, \mathscr{T}^p_{X^{\mathrm{an}}}) \otimes_{\mathbb{Q}} \mathbb{R}.\end{aligned}$$

□

4.2 The tropical cycle class map

In this section we will finally define the tropical cycle class map, which relates the Chow groups of a smooth variety X to its tropical Dolbeault cohomology groups.

Definition 4.2.1. Let X be a variety over K. For $p, q \geq 0$, we define the *tropical Dolbeault cohomology group* of X as

$$\mathrm{H}^{p,q}_{\mathrm{trop}}(X) := \mathrm{H}^q(X^{\mathrm{an}}, \mathscr{T}^p_{X^{\mathrm{an}}}),$$

which we regard by Corollary 4.1.9 as a rational subspace of $\mathrm{H}^{p,q}(X^{\mathrm{an}})$.

Remark 4.2.2. Let $\pi\colon X^{\mathrm{an}} \to X$ be the usual analytification morphism from Remark 3.3.1.

i.) The continuous map π induces naturally a morphism of sheaves $\mathscr{K}^p_X \to \pi_*\mathscr{K}^p_{X^{\mathrm{an}}}$ on X via $\mathcal{O}_X(U) \to \mathcal{O}_{X^{\mathrm{an}}}(U^{\mathrm{an}})$, where π_* denotes the direct image functor. Thus we obtain a morphism in cohomology

$$\mathrm{H}^\bullet(X, \mathscr{K}^p_X) \to \mathrm{H}^\bullet(X, \pi_*\mathscr{K}^p_{X^{\mathrm{an}}}). \tag{4.2.1}$$

ii.) Furthermore there is a natural map

$$\mathrm{H}^\bullet(X, \pi_*\mathscr{K}^p_{X^{\mathrm{an}}}) \to \mathrm{H}^\bullet(X^{\mathrm{an}}, \pi^{-1}\pi_*\mathscr{K}^p_{X^{\mathrm{an}}}),$$

where π^{-1} denotes the inverse image functor. This is purely sheaf-theoretic, for details see e.g. [Ive86, II.5.1]. By the adjunction of inverse and direct image, the counit morphism $\pi^{-1}\pi_*\mathscr{K}^p_{X^{\mathrm{an}}} \to \mathscr{K}^p_{X^{\mathrm{an}}}$ yields a morphism

$$\mathrm{H}^\bullet(X^{\mathrm{an}}, \pi^{-1}\pi_*\mathscr{K}^p_{X^{\mathrm{an}}}) \to \mathrm{H}^\bullet(X^{\mathrm{an}}, \mathscr{K}^p_{X^{\mathrm{an}}}).$$

Composing this with i.), we obtain a natural map in cohomology

$$\mathrm{H}^\bullet(X, \mathscr{K}^p_X) \to \mathrm{H}^\bullet(X^{\mathrm{an}}, \mathscr{K}^p_{X^{\mathrm{an}}}). \tag{4.2.2}$$

We can now finally define the tropical cycle class map.

Definition 4.2.3. Let X be a smooth variety over K. We define the *tropical cycle class map* $\mathrm{cl}_{\mathrm{trop}}$ as the composition

$$\mathrm{CH}^p(X)_{\mathbb{Q}} \xrightarrow{\mathrm{cl}_{\mathrm{univ}}} \mathrm{H}^p(X, \mathscr{K}_X^p) \xrightarrow{(4.2.2)} \mathrm{H}^p(X^{\mathrm{an}}, \mathscr{K}_{X^{\mathrm{an}}}^p) \xrightarrow{\mathrm{H}^p(X^{\mathrm{an}}, \tau_{X^{\mathrm{an}}}^p)} \mathrm{H}^p(X^{\mathrm{an}}, \mathscr{T}_{X^{\mathrm{an}}}^p) = \mathrm{H}_{\mathrm{trop}}^{p,p}(X).$$

Remark 4.2.4. Let X be a variety of dimension n and $\omega \in A_{X^{\mathrm{an}},c}^{n-p,n-p}(X^{\mathrm{an}})$ be a d''-closed differential form with compact support. Furthermore let $[\eta] \in \mathrm{H}^{p,p}(X^{\mathrm{an}})$ be given by a representative $\eta \in \ker(d''\colon A^{p,p}(X^{\mathrm{an}}) \to A^{p,p+1}(X^{\mathrm{an}}))$. Then there is a well-defined integral

$$\int_{X^{\mathrm{an}}} [\eta] \wedge \omega := \int_{X^{\mathrm{an}}} \eta \wedge \omega. \tag{4.2.3}$$

Clearly $\eta \wedge \omega$ has compact support as well. Furthermore (4.2.3) is independent of the choice of η. Indeed, let $\mu \in A^{p,p-1}(X^{\mathrm{an}})$ and we need to show that $\int_{X^{\mathrm{an}}} \eta \wedge \omega = \int_{X^{\mathrm{an}}} (\eta + d''\mu) \wedge \omega$. We have

$$\int_{X^{\mathrm{an}}} d''\mu \wedge \omega \overset{3.1.4}{=} \left(\int_{X^{\mathrm{an}}} d''(\mu \wedge \omega) \right) - (-1)^{2p-1} \left(\int_{X^{\mathrm{an}}} \mu \wedge d''\omega \right) = 0,$$

because $\int_{X^{\mathrm{an}}} d''(\mu \wedge \omega) = 0$ by Stokes' formula 3.4.20 and $\int_{X^{\mathrm{an}}} \mu \wedge d''\omega = 0$ as $d''\omega = 0$ by assumption. Hence (4.2.3) is well-defined.

We conclude with the following theorem by Liu, which can be regarded as a tropical version of the Cauchy formula in multi-variable complex analysis.

Theorem 4.2.5. Let X be a smooth variety over K of dimension n. For every algebraic cycle Z of X of codimension p, and any d''-closed differential form $\omega \in A_{X^{\mathrm{an}},c}^{n-p,n-p}(X^{\mathrm{an}})$ with compact support, we have the equality

$$\int_{X^{\mathrm{an}}} \mathrm{cl}_{\mathrm{trop}}(Z) \wedge \omega = \int_{Z^{\mathrm{an}}} \omega.$$

Here, if we formally write $Z = \sum_{i=1}^k a_i Z_i$ with $a_i \in \mathbb{Z}$ and the Z_i's closed subvarieties of codimension p, we define

$$\int_{Z^{\mathrm{an}}} \omega := \sum_{i=1}^{k} a_i \int_{Z_i^{\mathrm{an}}} \omega. \tag{4.2.4}$$

Proof. See [Liu17, Theorem 3.7]. □

Before we say a few words on how our discussions in Chapter 2 play a fundamental role in the proof of the theorem, we state an interesting corollary.

Corollary 4.2.6. Let X be a proper smooth variety over K of dimension n. Let Z be an algebraic cycle of X of dimension 0. Then we have

$$\int_{X^{\mathrm{an}}} \mathrm{cl}_{\mathrm{trop}}(Z) = \deg(Z).$$

Recall that the degree of Z is defined as $\deg(Z) = \sum_{i=1}^{k} a_i \in \mathbb{Z}$ for $Z = \sum_{i=1}^{k} a_i Z_i$.

Proof. By linearity we can assume that Z is prime, i.e. a closed subvariety of dimension 0. As X is proper, the analytification X^{an} is compact (see Remark 3.3.1). In particular we can choose ω to be the constant function 1 on X^{an} in Theorem 4.2.5, and we obtain $\int_{X^{\mathrm{an}}} \mathrm{cl}_{\mathrm{trop}}(Z) = \int_{Z^{\mathrm{an}}} 1$. A very affine chart of integration $U \subseteq Z$ is zero dimensional as well, hence $\mathrm{Trop}(U)$ consists of a single point. Integration is defined via pullbacks, and as we are integrating the constant function 1 in a zero-dimensional space, this is exactly 1. Hence

$$\int_{Z^{\mathrm{an}}} 1 = 1$$

for the prime cycle Z, and the claim follows by linearity. $\square$

Remark 4.2.7. We de not prove Theorem 4.2.5, however we want to conclude this thesis with an explanation of how the notion in the special description of $\mathrm{cl}_{\mathrm{univ}}$ in Section 2.4 is used in Liu's proof of the theorem. To explicitly describe the image of a cycle under $\mathrm{cl}_{\mathrm{trop}}$ via the composition in (4.2.3) in a 'naive way' is very hard. However we can utilize the description via regular sequences as in Construction 2.4.1 as follows:

Let Z be an algebraic cycle of X of codimension p. By linearity of the integral as defined in (4.2.4) we may assume that Z is prime, i.e. a closed subvariety of X of codimension p. Denote by $Z_{\mathrm{sing}} \subsetneq Z$ the singular locus of Z, which is a closed subscheme of X (with its induced reduced subscheme structure) of codimension $> p$. Furthermore put $U := X \setminus Z_{\mathrm{sing}}$ and $Z_{\mathrm{sm}} := Z \setminus Z_{\mathrm{sing}}$, which is a smooth closed subvariety of U of codimension p. Furthermore note that we have a closed immersion $Z_{\mathrm{sm}}^{\mathrm{an}} \hookrightarrow U^{\mathrm{an}}$.

For ease of notation, we put

$$A_{X^{\mathrm{an}}}^{p,q,\mathrm{cl}} := \ker(d'' \colon A_{X^{\mathrm{an}}}^{p,q} \to A_{X^{\mathrm{an}}}^{p,q+1})$$

for a variety X. Fix a form $\omega \in A_{X^{\mathrm{an}},c}^{n-p,n-p,\mathrm{cl}}(X^{\mathrm{an}})$, and by Corollary 3.4.14, ω belongs to $A_{X^{\mathrm{an}},c}^{n-p,n-p,\mathrm{cl}}(U^{\mathrm{an}})$. So the proof of the theorem reduces to U, Z_{sm} and examining the map $\mathrm{CH}^p(U)_{\mathbb{Q}} \to \mathrm{H}_{\mathrm{trop}}^{p,p}(U)$. Note that now we are in the situation of Construction 2.4.1, i.e. U is a smooth variety of dimension n, and Z_{sm} is a smooth closed subvariety of codimension p.

We want to utilize the fact that for a topological space Y with open cover $\mathfrak{U}$ and sheaf $\mathcal{F}$, the canonical morphism from Čech cohomology to sheaf cohomology in Remark B.7 is functorial in $\mathcal{F}$, i.e. that the diagram

$$\begin{tikzcd} \check{\mathrm{H}}^p(\mathfrak{U},\mathcal{F}) \arrow[r] \arrow[d] & \mathrm{H}^p(Y,\mathcal{F}) \arrow[d] \\ \check{\mathrm{H}}^p(\mathfrak{U},\mathcal{G}) \arrow[r] & \mathrm{H}^p(Y,\mathcal{G}) \end{tikzcd} \tag{4.2.5}$$

commutes for a sheaf morphism $\mathcal{F} \to \mathcal{G}$ on Y.

As in Construction 2.4.1 we choose a finite affine open covering U_α of U and regular sequences $f_{\alpha 1}, \dots, f_{\alpha p} \in \mathcal{O}_U(U_\alpha)$, such that $Z_{\mathrm{sm}} \cap U_\alpha$ is defined by the ideal $(f_{\alpha 1}, \dots, f_{\alpha p})$. Again we write $U_{\alpha i}$ for the nonvanishing locus $D(f_{\alpha i}) \subseteq U_\alpha$ of $f_{\alpha i}$. Then $\{U_{\alpha i} \mid 1 \le i \le p\}$ (respectively $\{U_{\alpha i}^{\mathrm{an}} \mid 1 \le i \le p\}$) forms an open covering of $U_\alpha \setminus Z_{\mathrm{sm}}$ (respectively $U_\alpha^{\mathrm{an}} \setminus Z_{\mathrm{sm}}^{\mathrm{an}}$) which we denote by $\mathfrak{U}_\alpha$ (respectively $\mathfrak{U}_\alpha^{\mathrm{an}}$). Note here that we have $(U_\alpha \setminus Z_{\mathrm{sm}})^{\mathrm{an}} = U_\alpha^{\mathrm{an}} \setminus Z_{\mathrm{sm}}^{\mathrm{an}}$. We obtain a commutative diagram

$$\begin{tikzcd} \check{\mathrm{H}}^{p-1}(\mathfrak{U}_\alpha, \mathscr{K}^p_{U_\alpha \setminus Z_{\mathrm{sm}}}) \arrow[r] \arrow[d, "\beta"'] & \mathrm{H}^{p-1}(U_\alpha \setminus Z_{\mathrm{sm}}, \mathscr{K}^p_{U_\alpha \setminus Z_{\mathrm{sm}}}) \arrow[d] \\ \check{\mathrm{H}}^{p-1}(\mathfrak{U}^{\mathrm{an}}_\alpha, \mathscr{K}^p_{U^{\mathrm{an}}_\alpha \setminus Z^{\mathrm{an}}_{\mathrm{sm}}}) \arrow[r] \arrow[d, "\tau^p_{U^{\mathrm{an}}}"'] & \mathrm{H}^{p-1}(U^{\mathrm{an}}_\alpha \setminus Z^{\mathrm{an}}_{\mathrm{sm}}, \mathscr{K}^p_{U^{\mathrm{an}}_\alpha \setminus Z^{\mathrm{an}}_{\mathrm{sm}}}) \arrow[d] \\ \check{\mathrm{H}}^{p-1}(\mathfrak{U}^{\mathrm{an}}_\alpha, A^{p,0,\mathrm{cl}}_{U^{\mathrm{an}}}) \arrow[r] & \mathrm{H}^{p-1}(U^{\mathrm{an}}_\alpha \setminus Z^{\mathrm{an}}_{\mathrm{sm}}, A^{p,0,\mathrm{cl}}_{U^{\mathrm{an}}}). \end{tikzcd} \tag{4.2.6}$$

The commutativity of the lower square is the commutativity in (4.2.5), and the upper square commutes by Remark 4.2.2 and as β is induced by the injective morphism $\mathcal{O}_U \to \pi_* \mathcal{O}_{U^{\mathrm{an}}}$.

Note that $\mathrm{H}^{p-1}(U^{\mathrm{an}}_\alpha \setminus Z^{\mathrm{an}}_{\mathrm{sm}}, A^{p,0,\mathrm{cl}}_{U^{\mathrm{an}}}) \cong \mathrm{H}^{p-1}(\Gamma(U^{\mathrm{an}}_\alpha \setminus Z^{\mathrm{an}}_{\mathrm{sm}}, A^{p,\bullet}_{U^{\mathrm{an}}}))$, in particular $\tau^p_{U^{\mathrm{an}}}(\{f_{\alpha 1}, \dots, f_{\alpha p}\})$ induces a cohomology class $[\theta_\alpha] \in \mathrm{H}^{p-1}(U^{\mathrm{an}}_\alpha \setminus Z^{\mathrm{an}}_{\mathrm{sm}}, A^{p,0,\mathrm{cl}}_{U^{\mathrm{an}}})$ given by a Dolbeault representative $\theta_\alpha \in A^{p,p-1,\mathrm{cl}}_{U^{\mathrm{an}}}(U^{\mathrm{an}}_\alpha \setminus Z^{\mathrm{an}}_{\mathrm{sm}})$.

By partition of unity we may write $\omega = \sum_\alpha \omega_\alpha$ with $\omega_\alpha \in A^{n-p,n-p}_{U^{\mathrm{an}},c}(U_\alpha)$. Then one can show (see Step 2 and Step 3 in the proof of [Liu17, Theorem 3.7]) that the theorem follows if

$$\int_{(U_\alpha \setminus Z_{\mathrm{sm}})^{\mathrm{an}}} \theta_\alpha \wedge d''\omega_\alpha = \int_{(U_\alpha \cap Z_{\mathrm{sm}})^{\mathrm{an}}} \omega_\alpha$$

holds for any α.

By construction and Proposition 2.4.2 this expression does neither depend on the choice of representative θ_α nor on the regular sequence $f_{\alpha 1}, \dots, f_{\alpha p}$. As the sheaves $A^{p,\bullet}_{U^{\mathrm{an}}}$ are fine, they are Čech-acyclic and we can explicitly construct θ_α from $\tau^p_{U^{\mathrm{an}}}(\{f_{\alpha 1}, \dots, f_{\alpha p}\})$ with the same strategy as in Remark B.7 and as in the proof of Proposition 2.4.2.

Appendix A

Sheaf Cohomology with Supports

We want to introduce subsheaves with support and their cohomology. This section basically forms explanations to Exercise II.1.20 and Exercise III.2.3 in [Har77].

We denote by (ab) the category of abelian groups, and by $\mathrm{Sh}_{ab}(X)$ the category of abelian sheaves on X.

Definition A.1. In the following let X always be a topological space, $Z \subseteq X$ a closed subset and $\mathcal{F}$ an abelian sheaf on X. We define $\Gamma_Z(X, \mathcal{F})$ to be the subgroup of $\mathcal{F}(X) = \Gamma(X, \mathcal{F})$ consisting of all sections whose support is contained in Z.

Lemma A.2. [Har77, Exercise II.1.20]

i.) The presheaf $V \to \Gamma_{Z\cap V}(V, \mathcal{F}|_V)$ is a sheaf. It is called the *subsheaf of* $\mathcal{F}$ *with supports in* Z, and is denoted by $\mathcal{H}^0_Z(\mathcal{F})$.

ii.) Let $U = X \setminus Z$ and let $j\colon U \to X$ denote the inclusion. There is an exact sequence of sheaves on X
$$0 \to \mathcal{H}^0_Z(\mathcal{F}) \to \mathcal{F} \to j_*(\mathcal{F}|_U).$$
If $\mathcal{F}$ is flasque, then the map $\mathcal{F} \to j_*(\mathcal{F}|_U)$ is surjective.

Proof. It is clear that $\mathcal{H} := \mathcal{H}^0_Z(\mathcal{F})$ is a presheaf. For i.) let $\bigcup_i V_i = V$ be an open cover of an open subset V. The injectivity of $\mathcal{H}(V) \to \prod_i \mathcal{H}(V_i)$ follows as $\mathcal{F}$ is a sheaf. If $(s_i \in \mathcal{H}(V_i))_{i\in I}$ are sections agreeing on intersections, they uniquely glue to $s \in \mathcal{F}(V)$, and $\mathrm{supp}(s) \subseteq Z \cap V$, as $s_x = (s_i)_x$ for $x \in V_i$. Hence $\mathcal{H}$ is a sheaf.

Show ii.): Clearly $\mathcal{H} = \ker(\mathcal{F} \to j_*(\mathcal{F}|_U)$, as for open $V \subseteq X$ we have $j_*(\mathcal{F}|_U) = \mathcal{F}(U\cap V) = \mathcal{F}(V \setminus Z)$. The surjectivity in the flasque case follows from the surjectivity of the restriction maps of $\mathcal{F}$. $\square$

Lemma A.3. [Har77, Exercise III.2.3 a)] The assignment

$$\Gamma_Z(X,\cdot)\colon \mathrm{Sh}_{ab}(X) \to (ab)$$

is a left exact functor.

Proof. Let $\varphi\colon \mathcal{F} \to \mathcal{G}$ be a map of sheaves and $U \subseteq X$ open. Furthermore let $s \in \Gamma_{U\cap Z}(U,\mathcal{F})$ and $x \in U \setminus Z$. Then $\varphi(U)(s)_x = \varphi_x(s_x) = \varphi_x(0) = 0$. Thus $\Gamma_{U\cap Z}(U,\mathcal{F}) \to \Gamma_{U\cap Z}(U,\mathcal{G})$ is well-defined and $\mathcal{H}^0_Z(\cdot)\colon \mathrm{Sh}_{ab}(X) \to \mathrm{Sh}_{ab}(X)$ is a functor. In particular $\Gamma_Z(X,\cdot)$ is a functor as composition of $\mathcal{H}^0_Z$ with the global sections functor $\Gamma(X,\cdot)$.

To see that $\Gamma_Z(X,\cdot)$ is left-exact, let

$$0 \to \mathcal{F}' \xrightarrow{\tilde{\alpha}} \mathcal{F} \xrightarrow{\tilde{\beta}} \mathcal{F}'' \to 0$$

be a short exact sequence of sheaves, and we need to show that the induced sequence of abelian groups

$$0 \to \Gamma_Z(X,\mathcal{F}') \xrightarrow{\alpha} \Gamma_Z(X,\mathcal{F}) \xrightarrow{\beta} \Gamma_Z(X,\mathcal{F}'')$$

is exact, where $\alpha = \tilde{\alpha}(X), \beta = \tilde{\beta}(X)$. The injectivity of α is clear. To see that $\ker(\beta) = \mathrm{im}(\alpha)$ let $s \in \Gamma_Z(X,\mathcal{F})$ with $\beta(s) = 0$. By the left-exactness of the global sections functor there is $t \in \Gamma(X,\mathcal{F}')$ with $\alpha(t) = s$. For any $x \notin Z$ we have on germs $\alpha_x(t_x) = s_x = 0$, hence by injectivity of α_x have $t_x = 0$. Thus $t \in \Gamma_Z(X,\mathcal{F}')$ and the lemma is proved. □

Definition A.4. We denote the right derived functors of the left exact functor $\Gamma_Z(X,\cdot)$ by $\mathrm{H}^i_Z(X,\cdot)$. They are called the *cohomology groups of* X *with supports in* Z, and coefficients in a given sheaf.

Proposition A.5. [Har77, Exercise III.2.3 b)-d)]

i.) If $0 \to \mathcal{F}' \xrightarrow{\tilde{\alpha}} \mathcal{F} \xrightarrow{\tilde{\beta}} \mathcal{F}'' \to 0$ is an exact sequence of sheaves, with $\mathcal{F}'$ flasque, then

$$0 \to \Gamma_Z(X,\mathcal{F}') \xrightarrow{\alpha} \Gamma_Z(X,\mathcal{F}) \xrightarrow{\beta} \Gamma_Z(X,\mathcal{F}'') \to 0$$

is exact. Here again $\alpha = \tilde{\alpha}(X), \beta = \tilde{\beta}(X)$.

ii.) If $\mathcal{F}$ is flasque, then $\mathrm{H}^i_Z(X,\mathcal{F}) = 0$ for all $i > 0$.

iii.) Let $U := X \setminus Z$. If $\mathcal{F}$ is flasque, then the sequence

$$0 \to \Gamma_Z(X,\mathcal{F}) \to \Gamma(X,\mathcal{F}) \to \Gamma(U,\mathcal{F}) \to 0$$

is exact.

Proof. i.) By left-exactness of $\Gamma_Z(X,\cdot)$, it remains only to show that $\Gamma_Z(X,\mathcal{F}) \xrightarrow{\beta} \Gamma_Z(X,\mathcal{F}'')$ is surjective. As $\mathcal{F}'$ is flasque, it is known that (see e.g. [Har77, Exercise II.1.16]) the sequence

$$0 \to \Gamma(X,\mathcal{F}') \xrightarrow{\alpha} \Gamma(X,\mathcal{F}) \xrightarrow{\beta} \Gamma(X,\mathcal{F}'') \to 0$$

is exact, hence any $s'' \in \Gamma_Z(X,\mathcal{F}'')$ lifts to some $s \in \Gamma(X,\mathcal{F})$ with $\beta(s) = s''$. For any $x \in U := X \setminus Z$ there is an open $V \subseteq U$ around x with $s''|_V = 0$, so $\tilde{\beta}(V)(s|_V) = \beta(s)|_V = s''|_V = 0$. By exactness of the original sequence of sheaves, there is an open $U_x \subseteq U$ around x and $t^x \in \mathcal{F}'(U_x)$ with $\tilde{\alpha}(U_x)(t^x) = s|_{U_x}$. Due to $\tilde{\alpha}(U_x)(t^x)|_{U_x\cap U_y} = s|_{U_x\cap U_y} = \tilde{\alpha}(U_y)(t^y)|_{U_x\cap U_y}$ for all $x,y \in U$ and injectivity of $\tilde{\alpha}$, the t^x glue to some $t^U \in \mathcal{F}'(U)$, which again – by flasqueness of $\mathcal{F}'$ – can be extended to some $t \in \Gamma(X,\mathcal{F}')$. Consider now the element $\tilde{s} := s - \alpha(t) \in \Gamma(X,\mathcal{F})$. Then

$$\beta(\tilde{s}) = \beta(s) - \underbrace{\beta\circ\alpha}_{=0}(t) = s''$$

and $\tilde{s} \in \Gamma_Z(X,\mathcal{F})$, as $\tilde{s}|_V = s|_V - \underbrace{\tilde{\alpha}(V)(t|_V)}_{=s|_V} = 0$. This shows i.).

Obtain ii.) in exactly the same way as in the proof of [Har77, Proposition III.2.5].

To show iii.), note that by Lemma A.2 i.) we have an exact sequence of sheaves

$$0 \to \mathcal{H}^0_Z(\mathcal{F}) \to \mathcal{F} \to j_*(\mathcal{F}|_U) \to 0,$$

where $j\colon U \to X$ denotes the inclusion. We apply the left exact functor $\Gamma(X,\cdot)$ to obtain the left exact short sequence

$$0 \to \Gamma_Z(X,\mathcal{F}) \to \Gamma(X,\mathcal{F}) \to \Gamma(U,\mathcal{F}) \to 0,$$

which is exact, as $\mathcal{F}$ is flasque. $\square$

Corollary A.6. (Long exact sequence of cohomology with supports) [Har77, Exercise III.2.3 e)]

Let X be a topological space, $Z \subseteq X$ closed and $U := X \setminus Z$. Then for any sheaf of abelian groups $\mathcal{F}$, there is a long exact sequence of cohomology groups

$$\begin{aligned} 0 &\to \mathrm{H}^0_Z(X,\mathcal{F}) \to \mathrm{H}^0(X,\mathcal{F}) \to \mathrm{H}^0(U,\mathcal{F}|_U) \to \\ &\to \mathrm{H}^1_Z(X,\mathcal{F}) \to \mathrm{H}^1(X,\mathcal{F}) \to \mathrm{H}^1(U,\mathcal{F}|_U) \to \\ &\to \mathrm{H}^2_Z(X,\mathcal{F}) \to \cdots . \end{aligned} \tag{A.1}$$

Proof. We fix an injective resolution $\mathcal{I}^\bullet$ of $\mathcal{F}$, from which we obtain a flasque resolution $\mathcal{I}|_U^\bullet$ of $\mathcal{F}|_U$. Now Proposition A.5 iii.) gives a short exact sequence of complexes of abelian groups

$$0 \to \Gamma_Z(X, \mathcal{I}^\bullet) \to \Gamma(X, \mathcal{I}^\bullet) \to \Gamma(U, \mathcal{I}|_U^\bullet) \to 0,$$

whose induced long exact sequence in group cohomology is the sequence desired. Observe here that for direct computations we only require the resolution to be flasque. $\square$

Appendix B

Čech Cohomology

We review the basic notions of Čech cohomology groups with respect to a sheaf $\mathcal{F}$ of abelian groups. We are particularly interested in the canonical homomorphism from Čech cohomology to sheaf cohomology, which we will define below.

Definition B.1. (See [Har77, Section III.4]) Let X be a topological space and let $\mathfrak{U} = \{U_i\}_{i\in I}$ be an open covering of X. We fix once and for all a well-ordering of the index set I. For any finite set of indices $i_0, \dots, i_p \in I$ we denote the intersection $U_{i_0} \cap \dots \cap U_{i_p}$ by $U_{i_0,\dots,i_p}$. Furthermore let $\mathcal{F}$ be an abelian sheaf on X.

i.) We define the *Čech complex* $C^\bullet(\mathfrak{U}, \mathcal{F})$ of abelian groups as follows. For each $p \geq 0$, let

$$C^p(\mathfrak{U}, \mathcal{F}) = \prod_{i_0 < \dots < i_p \in I} \mathcal{F}(U_{i_0,\dots,i_p}).$$

Hence an element $\alpha \in C^p(\mathfrak{U}, \mathcal{F})$ is determined by giving an element

$$\alpha_{i_0,\dots,i_p} \in \mathcal{F}(U_{i_0,\dots,i_p})$$

for each ordered $(p+1)$-tuple $i_0 < \dots < i_p$ of elements of I. Define a coboundary map $\delta^p \colon C^p(\mathfrak{U}, \mathcal{F}) \to C^{p+1}(\mathfrak{U}, \mathcal{F})$ by setting

$$(\delta\alpha)_{i_0,\dots,i_{p+1}} = \sum_{k=0}^{p+1} (-1)^k \alpha_{i_0,\dots,\hat{i}_k,\dots,i_{p+1}}|_{U_{i_0,\dots,i_{p+1}}}$$

for $i_0 < \dots < i_{p+1} \in I$. Here the notation $\hat{i}_k$ means to omit i_k. As always we often leave out the exponent in the coboundary maps for ease of notation. We check easily that $\delta \circ \delta = 0$, so $C^\bullet(\mathfrak{U}, \mathcal{F})$ is indeed a complex.

ii.) We define the p-th *Čech cohomology group of $\mathcal{F}$, with respect to the covering* $\mathfrak{U}$, to be the p-th group cohomology

$$\check{\mathrm{H}}^p(\mathfrak{U}, \mathcal{F}) := \mathrm{H}^p(C^\bullet(\mathfrak{U}, \mathcal{F})).$$

Remark B.2.

i.) ([Har77, Lemma III.4.1]) Note that by the glueing property of the sheaf $\mathcal{F}$ we always have

$$\check{\mathrm{H}}^0(\mathfrak{U}, \mathcal{F}) \cong \Gamma(X, \mathcal{F}).$$

ii.) Let $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ be a short exact sequence of *presheaves* on X, i.e. for every open $U \subseteq X$ the sequence of abelian groups $0 \to \mathcal{F}'(U) \to \mathcal{F}(U) \to \mathcal{F}''(U) \to 0$ is exact. Then for any open covering $\mathfrak{U}$ of X we obtain a short exact sequence of complexes

$$0 \to C^\bullet(\mathfrak{U}, \mathcal{F}') \to C^\bullet(\mathfrak{U}, \mathcal{F}) \to C^\bullet(\mathfrak{U}, \mathcal{F}'') \to 0,$$

and thus there is a long exact sequence in Čech cohomology

$$\cdots \to \check{\mathrm{H}}^p(\mathfrak{U}, \mathcal{F}') \to \check{\mathrm{H}}^p(\mathfrak{U}, \mathcal{F}) \to \check{\mathrm{H}}^p(\mathfrak{U}, \mathcal{F}'') \to \check{\mathrm{H}}^{p+1}(\mathfrak{U}, \mathcal{F}) \to \cdots.$$

iii.) A short exact sequence of sheaves on a space X does not have to give a long exact sequence in Čech cohomology though. For example if we define an open covering $\mathfrak{U}$ to consist merely of X, then we see that $\check{\mathrm{H}}^p(\mathfrak{U}, \mathcal{F}) = 0$ for $p > 0$ and any sheaf $\mathcal{F}$. Hence with i.) a long exact sequence in Čech cohomology would imply that any short exact sequence of sheaves $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ gives rise to an exact sequence $0 \to \Gamma(X, \mathcal{F}') \to \Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{F}'') \to 0$. This would imply that the global sections functor is exact.

Definition B.3. The complex $C^\bullet(\mathfrak{U}, \mathcal{F})$ is a complex of abelian groups, yet we can also define a 'sheafified' version. Define for any $p \geq 0$ a sheaf on X by

$$\mathscr{C}^p(\mathfrak{U}, \mathcal{F}) := \prod_{i_0 < \cdots < i_p \in I} i_*(\mathcal{F}|_{U_{i_0, \ldots, i_p}}),$$

where $i \colon U_{i_0, \ldots, i_p} \hookrightarrow X$ denotes the respective inclusion map. As above, we obtain a corresponding coboundary morphism $\delta \colon \mathscr{C}^p(\mathfrak{U}, \mathcal{F}) \to \mathscr{C}^{p+1}(\mathfrak{U}, \mathcal{F})$ and in particular a complex of sheaves $(\mathscr{C}^\bullet(\mathfrak{U}, \mathcal{F}), \delta)$.

Note that by construction we have $\Gamma(X, \mathscr{C}^p(\mathfrak{U}, \mathcal{F})) = C^p(\mathfrak{U}, \mathcal{F})$ for each p.

Lemma B.4. The canonical sheaf morphism $\epsilon\colon \mathcal{F} \to \mathscr{C}^0(\mathfrak{U}, \mathcal{F})$, given by taking the product of the natural morphisms $\mathcal{F} \to i_*\mathcal{F}|_{U_j}$ for $j \in I$, gives rise to an exact sequence of sheaves

$$0 \to \mathcal{F} \xrightarrow{\epsilon} \mathscr{C}^0(\mathfrak{U}, \mathcal{F}) \xrightarrow{\delta} \mathscr{C}^1(\mathfrak{U}, \mathcal{F}) \xrightarrow{\delta} \cdots .$$

Proof. See [Har77, Lemma III.4.2]. □

Corollary B.5. If $\mathcal{F}$ is flasque, then $\check{\mathrm{H}}^p(\mathfrak{U}, \mathcal{F}) = 0$ for all $p > 0$.

Proof. See [Har77, Lemma III.4.3]. □

Remark B.6. If $0 \to \mathcal{F} \xrightarrow{\epsilon'} \mathcal{I}^0 \to \mathcal{I}^1 \to \cdots$ is an injective resolution of $\mathcal{F}$ and $0 \to \mathcal{F} \xrightarrow{\epsilon} \mathscr{C}^0(\mathfrak{U}, \mathcal{F}) \to \mathscr{C}^1(\mathfrak{U}, \mathcal{F}) \to \cdots$ is the Čech resolution as above, then there is a morphism of complexes $\eta\colon \mathscr{C}^\bullet(\mathfrak{U}, \mathcal{F}) \to \mathcal{I}^\bullet$, unique up to homotopy, for which the diagram

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\epsilon"] \arrow[dr, "\epsilon'"'] & \mathscr{C}^\bullet(\mathfrak{U}, \mathcal{F}) \arrow[d, "\eta"] \\ & \mathcal{I}^\bullet \end{tikzcd}$$

commutes, and thus induces a unique homomorphism

$$\check{\mathrm{H}}^\bullet(\mathfrak{U}, \mathcal{F}) \to \mathrm{H}^\bullet(X, \mathcal{F}), \tag{B.1}$$

functorial in $\mathcal{F}$. Of course initially this homomorphism is in its nature abstract, however one can give an explicit description via the total complex of the double complex (B.1), as it is done in [KCD08, Lemma 3.5, Proposition 3.6].

Remark B.7. For our practical applications to Chow groups we construct a morphism

$$\check{\mathrm{H}}^\bullet(\mathfrak{U}, \mathcal{F}) \to \mathrm{H}^\bullet(X, \mathcal{F})$$

as follows, which we call *the canonical homomorphism from Čech cohomology to sheaf cohomology*. We stress that the construction of this morphism is – up to sign – exactly the explicit description of (B.1) as in [KCD08, Lemma 3.5]. It is functorial in $\mathcal{F}$, i.e. the diagram

$$\begin{tikzcd} \check{\mathrm{H}}^n(\mathfrak{U}, \mathcal{F}) \arrow[r] \arrow[d] & \mathrm{H}^n(X, \mathcal{F}) \arrow[d] \\ \check{\mathrm{H}}^n(\mathfrak{U}, \mathcal{G}) \arrow[r] & \mathrm{H}^n(X, \mathcal{G}) \end{tikzcd}$$

commutes for any sheaf morphism $\mathcal{F} \to \mathcal{G}$ on X and $n \geq 0$.

Now suppose we have an injective resolution (indeed it does not need to be injective; in fact we only need the sheaves $\mathcal{I}^\bullet$ to be both acyclic for Čech and sheaf cohomology)

$$0 \to \mathcal{F} \xrightarrow{d^{-1}} \mathcal{I}^0 \xrightarrow{d^0} \mathcal{I}^1 \xrightarrow{d^1} \cdots .$$

By functoriality of the Čech complex, we obtain a naive double complex of sheaves

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{F} \arrow[r, "\delta^{-1,-1}"] \arrow[d, "d^{-1,-1}"] & \mathscr{C}^0(\mathfrak{U},\mathcal{F}) \arrow[r, "\delta^{-1,0}"] \arrow[d, "d^{-1,0}"] & \mathscr{C}^1(\mathfrak{U},\mathcal{F}) \arrow[r, "\delta^{-1,1}"] \arrow[d, "d^{-1,1}"] & \cdots \\
0 \arrow[r] & \mathcal{I}^0 \arrow[r, "\delta^{0,-1}"] \arrow[d, "d^{0,-1}"] & \mathscr{C}^0(\mathfrak{U},\mathcal{I}^0) \arrow[r, "\delta^{0,0}"] \arrow[d, "d^{0,0}"] & \mathscr{C}^1(\mathfrak{U},\mathcal{I}^0) \arrow[r, "\delta^{0,1}"] \arrow[d, "d^{0,1}"] & \cdots . \\
0 \arrow[r] & \mathcal{I}^1 \arrow[r, "\delta^{1,-1}"] \arrow[d] & \mathscr{C}^0(\mathfrak{U},\mathcal{I}^1) \arrow[r, "\delta^{1,0}"] \arrow[d] & \mathscr{C}^1(\mathfrak{U},\mathcal{I}^1) \arrow[r, "\delta^{1,1}"] \arrow[d] & \cdots \\
 & \vdots & \vdots & \vdots &
\end{tikzcd} \tag{B.1}$$

'Naive' in this sense means that we require the squares only to commute, not anti-commute. Applying the global sections functor to (B.1) gives the naive double complex of abelian groups

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \Gamma(X,\mathcal{F}) \arrow[r, "\delta"] \arrow[d, "d"] & C^0(\mathfrak{U},\mathcal{F}) \arrow[r, "\delta"] \arrow[d, "d"] & C^1(\mathfrak{U},\mathcal{F}) \arrow[r, "\delta"] \arrow[d, "d"] \arrow[d, dashed, shift right] & \cdots \\
0 \arrow[r] & \Gamma(X,\mathcal{I}^0) \arrow[r, "\delta"] \arrow[d, "d"] & C^0(\mathfrak{U},\mathcal{I}^0) \arrow[r, "\delta"] \arrow[d, "d"] \arrow[d, dashed, shift right] & C^1(\mathfrak{U},\mathcal{I}^0) \arrow[r, "\delta"] \arrow[d, "d"] \arrow[ul, dashed, bend right] & \cdots , \\
0 \arrow[r] & \Gamma(X,\mathcal{I}^1) \arrow[r, "\delta"] \arrow[d] & C^0(\mathfrak{U},\mathcal{I}^1) \arrow[r, "\delta"] \arrow[d] \arrow[ul, dashed, bend right] & C^1(\mathfrak{U},\mathcal{I}^1) \arrow[r, "\delta"] \arrow[d] & \cdots \\
 & \vdots & \vdots & \vdots &
\end{tikzcd} \tag{B.2}$$

in which all rows except the first one are exact, as $\mathcal{I}^p$ is Čech-acyclic for $p \geq 0$. Now let $[\eta_0] \in \check{\mathrm{H}}^n(\mathfrak{U},\mathcal{F})$ be given by a cocycle $\eta_0 \in C^n(\mathfrak{U},\mathcal{F})$. We obtain the image of $[\eta_0]$ under the canonical morphism from Čech cohomology to sheaf cohomology as follows:

We have $\delta \circ d(\eta_0) = d \circ \delta(\eta_0) = 0$. By exactness of the rows, there exists $\eta_1 \in C^{n-1}(\mathfrak{U},\mathcal{I}^0)$ with $\delta(\eta_1) = d(\eta_0)$. It follows that $\delta(d\eta_1) = d(\delta\eta_1) = dd(\eta_0) = 0$, and there exists $\eta_2 \in C^{n-2}(\mathfrak{U},\mathcal{I}^1)$ with $\delta\eta_2 = d\eta_1$, and so forth. Eventually we obtain $\theta := \eta_{n+1} \in \Gamma(X,\mathcal{I}^n)$ with

$\delta\theta = d\eta_n$. Then $d\theta = 0$ since $\delta(d\theta) = d(\delta\theta) = dd(\eta_n) = 0$, and as the rows of the double complex are exact. Hence θ induces an element $[\theta] \in \mathrm{H}^n(X, \mathcal{F})$, which is our desired image.

This 'staircase walk' is indicated in the diagram (B.2) above with dashed arrows.

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