The Laplacian on a metrized graph

Jonathan Pirnay (MN 1740960)

Lecture in the Seminar on Potential Theory on the Berkovich Projective Line
on June 11th 2018

Throughout this document let Γ always denote a metrized graph with a fixed orientation.

1. The Laplacian on BDV(Γ)

1.1. **Reminder.**

i.) We have defined $\mathrm{CPA}(\Gamma) := \{f\colon \Gamma \to \mathbb{R} \mid f \text{ continuous, piecewise affine}\}$ and $\mathrm{Zh}(\Gamma)$ as the set of all continuous functions $f\colon \Gamma \to \mathbb{R}$ such that f is piecewise $\mathcal{C}^2$ (i.e. exists vertex set $X_f \subseteq \Gamma$ such that $\Gamma \setminus X_f$ is finite union of open intervals and restriction of f to each of those is $\mathcal{C}^2$) and $f''(x) \in L^1(\Gamma, dx)$.
Furthermore $\mathcal{D}(\Gamma) := \{f\colon \Gamma \to \mathbb{R} \mid d_{\vec{v}}f(p) \text{ exists } \forall p \in \Gamma, \vec{v} \in T_p(\Gamma)\}$ and Laplacian

$$\Delta_{\mathrm{Zh}} := -f''(x)dx + \sum_{p\in\Gamma}(-\sum_{\vec{v}\in T_p(\Gamma)} d_{\vec{v}}f(p))\delta_p(x). \tag{1}$$

ii.) Obviously $\mathrm{CPA}(\Gamma) \subseteq \mathrm{Zh}(\Gamma)$ and $\Delta_{\mathrm{Zh}}|_{\mathrm{CPA}(\Gamma)} = \Delta_{\mathrm{CPA}}$.

iii.) Let $\mathcal{A} := \mathcal{A}(\Gamma)$ be the Boolean algebra of subsets of Γ generated by the connected open sets. Each $S \in \mathcal{A}$ is a finite disjoint union of sets isometric to open, half-open or (possibly degenerate) closed intervals.

iv.) For $f \in \mathcal{D}(\Gamma)$ we have defined a finitely additive set function m_f on $\mathcal{A}$ by requiring that for each $S \in \mathcal{A}$ have

$$m_f(S) = \sum_{\substack{p\in b(S),\\ p\notin S}} \sum_{\vec{v}\in \mathrm{In}(p,S)} d_{\vec{v}}f(p) - \sum_{\substack{p\in b(S),\\ p\in S}} \sum_{\vec{v}\in \mathrm{Out}(p,S)} d_{\vec{v}}f(p). \tag{2}$$

Here $b(S) = \overline{S} \cap \overline{\Gamma \setminus S}$ as usual and for $p \in \Gamma$ define $\mathrm{In}(p,S)$ as the set of all $\vec{v} \in T_p(\Gamma)$ for which $p + t\vec{v}$ belongs to S for all sufficiently small $t > 0$. Accordingly $\mathrm{Out}(p,S) := T_p(\Gamma) \setminus \mathrm{In}(p,S)$.

v.) The linear subspace $\mathrm{BDV}(\Gamma) \subseteq \mathcal{D}(\Gamma)$ is defined as the set of functions $f \in \mathcal{D}(\Gamma)$ of *bounded differential variation*, i.e. there exists $B > 0$ such that for any countable family $\mathcal{F}$ of pairwise disjoint sets of $\mathcal{A}$ have

$$\sum_{S_i\in\mathcal{F}} |m_f(S_i)| \leq B. \tag{3}$$

vi.) For $f \in \mathrm{BDV}(\Gamma)$ the function m_f extends to a finite, signed Borel measure m_f^* of total mass 0 on Γ.

1.2. **Definition.** For $f \in \mathrm{BDV}(\Gamma)$ define the Laplacian $\Delta(f)$ as the finite, signed Borel measure

$$\Delta(f) := m_f^*.$$

1.3. **Lemma.** $\mathrm{Zh}(\Gamma) \subseteq \mathrm{BDV}(\Gamma)$ and for $f \in \mathrm{Zh}(\Gamma)$ have $\Delta(f) = \Delta_{\mathrm{Zh}}(f)$.

Proof: Let $f \in \mathrm{Zh}(\Gamma)$ and X_f a vertex set for Γ such that $f \in \mathcal{C}^2(\Gamma \setminus X_f)$. To see that $f \in \mathcal{D}(\Gamma)$ we need to show that $d_{\vec{v}}f(p)$ exists for all $p \in X_f$ and $\vec{v} \in T_p(\Gamma)$. Hence let $p \in X_f$ and $\vec{v} \in T_p(\Gamma)$ and let $t_0 > 0$ such that $p + t\vec{v} \in \Gamma \setminus X_f$ for all $t \in (0, t_0)$. Furthermore abuse notation by writing $f(t)$ for $f(p + t\vec{v})$ and observe that $f \in \mathcal{C}^2((0, t_0))$. Obviously $d_{\vec{v}}f(p)$ exists if and only if $\lim_{t \to 0^+} f'(t)$ exists. So let $\epsilon > 0$ and choose $0 < \delta < t_0$ in a way that $\int_{(0,\delta)} |f''(t)dt| < \epsilon$, which is possible as $f'' \in L^1(\Gamma, dx)$. Then for all $t_1, t_2 \in (0, \delta)$ we get

$$|f'(t_2) - f'(t_1)| = \left| \int_{t_1}^{t_2} f''(t)dt \right| \le \int_{t_1}^{t_2} |f''(t)|\, dt < \epsilon, \tag{4}$$

hence $\lim_{t \to 0^+} f'(t)$ exists and $f \in \mathcal{D}(\Gamma)$.

Now let $\{E_i\}_{i \in \mathbb{N}}$ be family of pairwise disjoint sets in $\mathcal{A}$. By [BR10, Prop. 3.5(B)] we can assume that $E_i \in \mathcal{A}$ is connected and closed $\forall i \in \mathbb{N}$, hence even further we may assume that $\{E_i\}_{i \in \mathbb{N}}$ consists of disjoint sets which are either a closed interval or an isolated point. For $p \in \Gamma \setminus X_f$ have $m_f(\{p\}) = -\sum_{\vec{v} \in T_p(\Gamma)} d_{\vec{v}}f(p) = 0$ as seen before, and for a closed interval $[t_0, t_1]$ on an edge of $\Gamma \setminus X_f$ obtain with (2) and (4)

$$|m_f([t_0, t_1])| = \left| \sum_{\substack{p \in b([t_0,t_1]),\\ p \in [t_0,t_1]}} \sum_{\vec{v} \in \mathrm{Out}(p,[t_0,t_1])} d_{\vec{v}}f(p) \right| = |f'(t_1) - f'(t_0)| \le \int_{t_0}^{t_1} |f''(t)|\, dt.$$

Using this obtain

$$\sum_{i \in \mathbb{N}} |m_f(E_i)| \le \sum_{p \in X_f} |m_f(\{p\})| + \int_\Gamma |f''(t)|\, dt < \infty,$$

hence $f \in \mathrm{BDV}(\Gamma)$ as desired.

It remains to show that $\Delta(f) = \Delta_{\mathrm{Zh}}(f)$. For this it suffices to show equality on points $p \in X_f$ and open intervals (c, d) contained in an edge of $\Gamma \setminus X_f$. For $p \in X_f$ get

$$\Delta(f)(\{p\}) = -\sum_{\vec{v} \in T_p(\Gamma)} d_{\vec{v}}f(p) = \Delta_{\mathrm{Zh}}(f)(\{p\}).$$

For (c,d) as above get by (2) that

$$\Delta(f)((c,d)) = m_f((c,d)) = \sum_{\substack{p\in b((c,d)),\\ p\notin(c,d)}} \sum_{\vec{v}\in \mathrm{In}(p,(c,d))} d_{\vec{v}}f(p)$$

$$= f'(c) - f'(d) = -\int_c^d f''(x)dx = \Delta_{\mathrm{Zh}}(f)((c,d)).$$

□

1.4. **Proposition.** Let $f \in \mathrm{BDV}(\Gamma)$ and assume

$$\Delta(f) = g(x)dx + \sum_{p_i\in X} c_{p_i}\delta_{p_i}(x) \tag{5}$$

for a piecewise continuous function $g \in L^1(\Gamma, dx)$ and a finite set $X \subseteq \Gamma$. Furthermore let $X_g \subseteq \Gamma$ be a vertex set containing X and the finitely many points where g is not continuous. Put $c_{p_i} := 0\ \forall p_i \in X_g \setminus X$. Then the following holds:

i.) $f''(x) = -g(x)\ \forall x \in \Gamma \setminus X_g$,
ii.) $f \in \mathrm{Zh}(\Gamma)$,
iii.) $\Delta(f)(\{p_i\}) = c_{p_i}\ \forall p_i \in X_g$.

Proof. Consider an edge in $\Gamma\setminus X_g$, identifying it with an interval (a,b) via our chosen parametrization. For each $x \in (a,b)$ have $-\sum_{\vec{v}\in T_x(\Gamma)} d_{\vec{v}}f(x) = \Delta(f)(\{x\}) = 0$, where the last equality follows from (5) as $x \notin X$, hence $f'(x)$ exists.
For small $h > 0$ get

$$\begin{aligned} f'(x+h) - f'(x) &= -(-(-f'(x) + f'(x+h))) \\ &= -(-(\sum_{\substack{p\in b([x,x+h]),\\ p\in[x,x+h]}} \sum_{\vec{v}\in \mathrm{Out}(p,[x,x+h])} d_{\vec{v}}f(p))) \\ &= -\Delta(f)([x,x+h]) = -\int_x^{x+h} g(t)dt. \end{aligned}$$

Analogously for $h < 0$ obtain $f'(x+h) - f'(x) = -(-f'(x+h) + f'(x)) = \Delta(f)([x+h,h]) = \int_{x+h}^{x} g(t)dt = -\int_x^{x+h} g(t)dt$. Hence

$$f''(x) = \lim_{h\to 0} \frac{f'(x+h) - f'(x)}{h} = \lim_{h\to 0}\left(-\frac{1}{h}\cdot \int_x^{x+h} g(t)dt\right) = -g(x),$$

which shows i.), while ii.) and iii.) are direct consequences. □

1.5. **Corollary.** If $f \in \mathrm{BDV}(\Gamma)$ and $\Delta(f) = \sum_{i=1}^k c_i\delta_{p_i}$ is a discrete measure, then $f \in \mathrm{CPA}(\Gamma)$.

Proof. Since $\Delta(f)$ discrete, obtain by 1.4 ii.) that $f \in \mathrm{Zh}(\Gamma)$ and hence $\Delta(f) = \Delta_{\mathrm{Zh}}(f)$. Fixing appropriate vertex set X for Γ we see by 1.4 i.) that $f''(x) =$

$-g(x) = 0$ on $\Gamma \setminus X$, so $f(x)$ is affine on each segment of $\Gamma \setminus X \implies f \in \mathrm{CPA}(\Gamma)$. □

2. Finite signed Borel measures on Γ

Our aim now is to show that every finite signed Borel measure on Γ of total mass 0 already is the Laplacian of some function in $\mathrm{BDV}(\Gamma)$. We first remind of some measure-theoretic statements.

2.1. **Reminder.**

i.) (Weak convergence) Let X be metric space with Borel σ-algebra Σ. We say that a sequence $\{\mu_n\}$ of Borel measures *converges weakly* to Borel measure μ if for every $f \in \mathcal{C}_{\mathrm{bd}}(X)$ have

$$\lim_{n\to\infty} \int_X f d\mu_n = \int_X f d\mu.$$

Analogously define weak convergence for signed Borel measures.

ii.) (Hahn decomposition) Let μ be finite signed measure on measurable space (X, Σ). There exist two measurable sets P, N such that

a.) $P \cup N = X$ and $P \cap N = \emptyset$,

b.) $\mu(E) \geq 0 \ \forall E \in \Sigma$ with $E \subseteq P$,

c.) $\mu(E) \leq 0 \ \forall E \in \Sigma$ with $E \subseteq N$.

We get (nonnegative) measures μ^+ and μ^- by $\mu^+(E) = \mu(P \cap E)$ and $\mu^-(E) = \mu(N \cap E) \ \forall E \in \Sigma$.

Both μ^+ and μ^- are finite (nonnegative) measures and satisfy

$$\mu = \mu^+ - \mu^-. \tag{6}$$

The measure $|\mu| = \mu^+ + \mu^-$ is the *variation* of μ and $|\mu|(X)$ is called the *total variation* of μ.

iii.) With (6) one can show a "triangle inequality"

$$\begin{aligned} \left|\int f d\mu\right| &= \left|\int f d\mu^+ - \int f d\mu^-\right| \\ &\leq \left|\int f d\mu^+\right| + \left|\int f d\mu^-\right| \\ &\leq \int |f| \, d\mu^+ + \int |f| \, d\mu^- \\ &= \int |f| \, d|\mu| \, . \end{aligned} \tag{7}$$

2.2. **Definition.** Let ν be finite signed Borel measure on Γ. A sequence $\{\nu_n\}_{n\in\mathbb{N}}$ of finite signed Borel measures converges *moderately well* to ν if:

(A) There is bound $B > 0$ such that $|\nu_n|(\Gamma) \leq B \ \forall n \in \mathbb{N}$.

(B) For each segment $D \subseteq \Gamma$ (open, closed, half-open) we have $\lim_{n\to\infty} \nu_n(D) = \nu(D)$.

2.3. **Remark.** Let ν and $\{\nu_n\}$ as in Def. 2.2.

i) As each set in $\mathcal{A}$ is finite disjoint union of segments, condition (B) implies that

$$\lim_{n\to\infty} \nu_n(S) = \nu(S)\ \forall S \in \mathcal{A}, \tag{8}$$

in particular $\lim_{n\to\infty} \nu_n(\Gamma) = \nu(\Gamma)$, and $|\nu|(\Gamma) \leq B$.

ii) By construction of appropriate step functions using characteristic functions of elements in $\mathcal{A}$, we obtain for $f \in \mathcal{C}_{\mathrm{bd}}(\Gamma)$ with (8) that $\{\nu_n\}$ converges weakly to ν.

iii) For any finite signed Borel measure ν on Γ there is a sequence of discrete signed measures which converges moderately well to ν. For details of the construction see [BR10, Section 3.6, p.63].

We can finally state our main proposition.

2.4. **Proposition.** Let ν be finite signed Borel measure on Γ. Fix $z \in \Gamma$ and put $h(x) = \int_\Gamma j_z(x,y)d\nu(y)$. Let $M = |\nu|(\Gamma)$ be the total variation of ν. Then:

i.) Have $h \in \mathrm{BDV}(\Gamma)$ and $\Delta(h) = \nu - \nu(\Gamma)\delta_z$.

ii.) For each $x \in \Gamma$ and each $\vec{v} \in T_x(\Gamma)$ have $|d_{\vec{v}}h(x)| \leq M$.

iii.) Let $\{\nu_n\}_{n\in\mathbb{N}}$ be any sequence of finite signed Borel measures which converges weakly to ν. For each $n \in \mathbb{N}$ put $h_n(x) = \int_\Gamma j_z(x,y)d\nu_n(y)$. Then $\{h_n\}_{n\in\mathbb{N}}$ converges pointwise to h on Γ and if there is $B \geq 0$ such that $|\nu_n|(\Gamma) \leq B$ for all $n \in \mathbb{N}$, the convergence is uniform.

iv.) If $\{\nu_n\}_{n\in\mathbb{N}}$ converges moderately well to ν, then for each $x \in \Gamma$ and $\vec{v} \in T_x(\Gamma)$,

$$\lim_{n\to\infty} d_{\vec{v}}h_n(x) = d_{\vec{v}}h(x). \tag{9}$$

2.5. **Remark.** Statement iv.) need not hold if $\{\nu_n\}_{n\in\mathbb{N}_{\geq 1}}$ merely converges weakly to ν. For example let $\Gamma = [0,1], z = 0, \nu = \delta_1 - \delta_0$ and let $\nu_n = \delta_1 - \delta_{\frac{1}{n}}$ for each $n \geq 1$. Then $\{\nu_n\}$ converges weakly to ν, but not moderately well, as for $(0,1] \subseteq \Gamma$ we have $\nu((0,1]) = 1 \neq 0 = \lim_{n\to\infty} \underbrace{\nu_n((0,1])}_{=0}$.

Use the explicit construction of $j_z(x,y)$ as in [BR10, Section 3.3, p.52] and obtain

$$h(x) = \underbrace{j_0(x,1)}_{=x} - \underbrace{j_0(x,0)}_{=0} = x$$

and analogously

$$h_n(x) = \underbrace{j_0(x,1)}_{=x} - \underbrace{j_0(x,\frac{1}{n})}_{\substack{=x \text{ if } x<1/n,\\ 1/n \text{ else}}} = \max(0, x - \frac{1}{n}).$$

However for the unique $\vec{v} \in T_0(\Gamma)$ we have $d_{\vec{v}}h(0) = 1$, while $d_{\vec{v}}h_n(0) = 0$ $\forall n \geq 1$.

2.6. **Corollary.** If ν is finite signed Borel measure on Γ with $\nu(\Gamma) = 0$, then there exists $h \in \mathrm{BDV}(\Gamma)$ such that $\Delta(h) = \nu$.

Proof. Follows immediately from 2.4, part i.). □

2.7. **Corollary.** Let ν be finite signed Borel measure on Γ, let $y \in \Gamma$ and consider

$$F_y(x) := j_\nu(x, y) := \int_\Gamma j_\xi(x, y) d\nu(\xi).$$

Then $F_y \in \mathrm{BDV}(\Gamma)$ satisfying $\Delta_x(F_y) = \nu(\Gamma)\delta_y - \nu$.

Proof. From [BR10, Prop. 3.3(A)] we see that for any $z \in \Gamma$,

$$\begin{aligned} F_y(x) &= \int_\Gamma j_\xi(x, y) d\nu(\xi) \\ &= \int_\Gamma j_z(x, y) - j_z(x, \xi) - \underbrace{j_z(y, \xi) + j_z(\xi, \xi) d\nu(\xi)}_{=:C<\infty,\text{ independent of } x} \\ &= \nu(\Gamma) j_z(x, y) - \int_\Gamma j_z(x, \xi) d\nu(\xi) - C. \end{aligned}$$

With 2.4 and as $\Delta_x j_z(x, y) = \delta_y - \delta_z$ obtain

$$\Delta_x(F_y) = \nu(\Gamma)(\delta_y - \delta_z) - (\nu - \nu(\Gamma)\delta_z) = \nu(\Gamma)\delta_y - \nu.$$

□

2.8. **Proof of Proposition 2.4.** Fix $z \in \Gamma$ and put $h(x) = \int_\Gamma j_z(x, y) d\nu(y)$. We first show that $h \in \mathcal{D}(\Gamma)$, i.e. need to show that $d_{\vec{v}}h(x)$ exists for each $x \in \Gamma$ and $\vec{v} \in T_x(\Gamma)$. Observe that for such $x, \vec{v}$ have

$$d_{\vec{v}}h(x) = \lim_{\tau \to 0^+} \int_\Gamma \frac{j_z(x + \tau\vec{v}, y) - j_z(x, y))}{\tau} d\nu(y), \tag{10}$$

provided the limit exists.

Let S be a vertex set for Γ and consider τ small enough that $x + \tau\vec{v}$ lies on the edge of $\Gamma \setminus (S \cup \{x, z\})$ in direction of $\vec{v}$. Let w.l.o.g. $e_\tau = (x, x + \tau\vec{v})$ be the open segment contained in that edge. By [BR10, Prop. 3.3(A)] the function $t \to j_z(t, y)$ is continuous in t and affine on edges of $\Gamma \setminus (S \cup \{y, z\})$ (in particular the slope is constant there). So for $y \notin e_\tau$ we have $(j_z(x + \tau\vec{v}, y) - j_z(x, y))/\tau = \partial_{x,\vec{v}} j_z(x, y)$. This implies that

$$\begin{aligned} &\int_\Gamma \frac{j_z(x + \tau\vec{v}, y) - j_z(x, y))}{\tau} d\nu(y) \\ &= \int_{\Gamma \setminus e_\tau} \partial_{x,\vec{v}} j_z(x, y) d\nu(y) + \int_{e_\tau} \frac{j_z(x + \tau\vec{v}, y) - j_z(x, y))}{\tau} d\nu(y). \end{aligned} \tag{11}$$

If $y \in e_\tau$, [BR10, Prop. 3.3(A)] gives $|(j_z(x+\tau\vec{v},y)-j_z(x,y))/\tau| \le 1$ as $\rho(x+\tau\vec{v},x)=\tau$. If $y \notin e_\tau$, [BR10, Prop. 3.3(D)] gives $|\partial_{x,\vec{v}}j_z(x,y)| \le 1$. Hence as $\tau \to 0^+$ the first integral in (11) converges to $\int_\Gamma \partial_{x,\vec{v}}j_z(x,y)d\nu(y)$, while the second one is bounded by $|\nu|(e_\tau)$ and hence converges to 0. Thus the limit in (10) exists and we obtain

$$d_{\vec{v}}h(x) = \int_\Gamma \partial_{x,\vec{v}}j_z(x,y)d\nu(y). \tag{12}$$

Using again that $|\partial_{x,\vec{v}}j_z(x,y)| \le 1\ \forall y \in \Gamma$, we at once get $|d_{\vec{v}}h(x)| \le |\nu|(\Gamma) = M$, which proves ii.).

Now let $\{\nu_n\}$ be any sequence of finite signed Borel measures converging weakly to ν and put $h_n(x) := \int_\Gamma j_z(x,y)d\nu_n(y)$. For each x the kernel $F_x(y) = j_z(x,y)$ is continuous in y, nonnegative and bounded by [BR10, Prop. 3.3(A)], so $\{h_n\}$ converges pointwise to h just by definition of weak convergence in Reminder 2.1 i.). Also by [BR10, Prop. 3.3(A)] we have $|j_z(x_1,y)-j_z(x_2,y)| \le \rho(x_1,x_2)\ \forall x_1,x_2 \in \Gamma$, so if there is bound B such that $|\nu_n|(\Gamma) \le B$ for all n, we obtain

$$|h_n(x_1)-h_n(x_2)| \overset{2.1\text{ iii.)}}{\le} \int_\Gamma |j_z(x_1,y)-j_z(x_2,y)|\, d\,|\nu|(y) \le B\cdot\rho(x_1,x_2),$$

and the functions h_n are all bounded by the same Lipschitz constant. As Γ is compact, by standard calculus the convergence of $\{h_n\}$ to h is uniform, which shows iii.).

For part iv.) assume that $\{\nu_n\}$ converges moderately well to ν. Let $x \in \Gamma$ and $\vec{v} \in T_x(\Gamma)$; we need to show

$$\lim_{n\to\infty} d_{\vec{v}}h_n(x) = d_{\vec{v}}h(x),$$

or equivalently using (12),

$$\lim_{n\to\infty}\int_\Gamma \partial_{x,\vec{v}}j_z(x,y)d\nu_n(y) = \int_\Gamma \partial_{x,\vec{v}}j_z(x,y)d\nu(y). \tag{13}$$

We don't give a full proof, just a short note: The difficulty is that $\partial_{x,\vec{v}}j_z(x,y)$ need not be continuous and ν and ν_n might have point masses. However the conditions of moderately well convergence allow us to construct appropriate step functions to show (13). For full details see [BR10, Section 3.6, pp.65-66].

It remains to show part i.). By Remark 2.3 iii.) we can choose sequence of discrete signed measures $\{\nu_n\}$ converging moderately well to ν. Use notation of h_n as above. By definition of m_{h_n} and m_h as in (2) and by (9) we see that each $S \in \mathcal{A}$ satisfies

$$\lim_{n\to\infty} m_{h_n}(S) = m_h(S).$$

For $n \in \mathbb{N}$ denote $\nu_n = \sum_{i\in\mathbb{N}} \lambda_{i,n}\delta_{c_{i,n}}$ for $\lambda_{i,n} \in \mathbb{R}, c_{i,n} \in \Gamma$. Then

$$h_n(x) = \int_\Gamma j_z(x,y)d\nu_n(y) = \sum_{i\in\mathbb{N}} \lambda_{i_n} j_z(x,c_{i,n})$$

and hence

$$\begin{aligned}\Delta(h_n) &= \sum_{i\in\mathbb{N}} \lambda_{i,n}(\delta_{c_{i,n}} - \delta_z)\\ &= \sum_{i\in\mathbb{N}} \lambda_{i,n}\delta_{c_{i,n}} - \delta_z \sum_{i\in\mathbb{N}} \lambda_{i,n}\\ &= \nu_n - \nu_n(\Gamma)\cdot\delta_z.\end{aligned}$$

So $m_{h_n}(S) = \Delta(h_n)(S) = \nu_n(S) - \nu_n(\Gamma)\delta_z(S)\ \forall S \in \mathcal{A}$. Passing to $n \to \infty$ yields $m_h(S) = \nu(S) - \nu(\Gamma)\delta_z(S)$. For countable family $\{S_i\}$ of disjoint sets in $\mathcal{A}$ it follows then that

$$\sum_{i\in\mathbb{N}} |m_h(S_i)| \leq 2\,|\nu|(\Gamma),$$

so indeed $h \in \mathrm{BDV}(\Gamma)$.
The signed measure $\Delta(h) = m_h^*$ attached to h is determined by its values on sets in $\mathcal{A}$, hence it must coincide with $\nu - \nu(\Gamma)\delta_z$. This finishes the proof. $\square$

References

[BR10] M. Baker and R. Rumely. *Potential Theory and Dynamics on the Berkovich Projective Line*, Mathematical Surveys and Monographs, 159. American Mathematical Society, Providence, RI, 2010.