

# The Laplacian on a metrized graph

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Throughout this document let  $\Gamma$  always denote a metrized graph with a fixed orientation.

## 1. THE LAPLACIAN ON $\text{BDV}(\Gamma)$

### 1.1. Reminder.

- i.) We have defined  $\text{CPA}(\Gamma) := \{f: \Gamma \rightarrow \mathbb{R} \mid f \text{ continuous, piecewise affine}\}$  and  $\text{Zh}(\Gamma)$  as the set of all continuous functions  $f: \Gamma \rightarrow \mathbb{R}$  such that  $f$  is piecewise  $\mathcal{C}^2$  (i.e. exists vertex set  $X_f \subseteq \Gamma$  such that  $\Gamma \setminus X_f$  is finite union of open intervals and restriction of  $f$  to each of those is  $\mathcal{C}^2$ ) and  $f''(x) \in L^1(\Gamma, dx)$ .  
Furthermore  $\mathcal{D}(\Gamma) := \{f: \Gamma \rightarrow \mathbb{R} \mid d_{\vec{v}}f(p) \text{ exists } \forall p \in \Gamma, \vec{v} \in T_p(\Gamma)\}$  and Laplacian

$$(1) \quad \Delta_{\text{Zh}} := -f''(x)dx + \sum_{p \in \Gamma} \left( - \sum_{\vec{v} \in T_p(\Gamma)} d_{\vec{v}}f(p) \right) \delta_p(x).$$

- ii.) Obviously  $\text{CPA}(\Gamma) \subseteq \text{Zh}(\Gamma)$  and  $\Delta_{\text{Zh}}|_{\text{CPA}(\Gamma)} = \Delta_{\text{CPA}}$ .  
iii.) Let  $\mathcal{A} := \mathcal{A}(\Gamma)$  be the Boolean algebra of subsets of  $\Gamma$  generated by the connected open sets. Each  $S \in \mathcal{A}$  is a finite disjoint union of sets isometric to open, half-open or (possibly degenerate) closed intervals.  
iv.) For  $f \in \mathcal{D}(\Gamma)$  we have defined a finitely additive set function  $m_f$  on  $\mathcal{A}$  by requiring that for each  $S \in \mathcal{A}$  have

$$(2) \quad m_f(S) = \sum_{\substack{p \in b(S), \\ p \notin S}} \sum_{\vec{v} \in \text{In}(p, S)} d_{\vec{v}}f(p) - \sum_{\substack{p \in b(S), \\ p \in S}} \sum_{\vec{v} \in \text{Out}(p, S)} d_{\vec{v}}f(p).$$

Here  $b(S) = \overline{S} \cap \overline{\Gamma \setminus S}$  as usual and for  $p \in \Gamma$  define  $\text{In}(p, S)$  as the set of all  $\vec{v} \in T_p(\Gamma)$  for which  $p + t\vec{v}$  belongs to  $S$  for all sufficiently small  $t > 0$ . Accordingly  $\text{Out}(p, S) := T_p(\Gamma) \setminus \text{In}(p, S)$ .

- v.) The linear subspace  $\text{BDV}(\Gamma) \subseteq \mathcal{D}(\Gamma)$  is defined as the set of functions  $f \in \mathcal{D}(\Gamma)$  of *bounded differential variation*, i.e. there exists  $B > 0$  such that for any countable family  $\mathcal{F}$  of pairwise disjoint sets of  $\mathcal{A}$  have

$$(3) \quad \sum_{S_i \in \mathcal{F}} |m_f(S_i)| \leq B.$$

- vi.) For  $f \in \text{BDV}(\Gamma)$  the function  $m_f$  extends to a finite, signed Borel measure  $m_f^*$  of total mass 0 on  $\Gamma$ .

**1.2. Definition.** For  $f \in \text{BDV}(\Gamma)$  define the Laplacian  $\Delta(f)$  as the finite, signed Borel measure

$$\Delta(f) := m_f^*.$$

**1.3. Lemma.**  $\text{Zh}(\Gamma) \subseteq \text{BDV}(\Gamma)$  and for  $f \in \text{Zh}(\Gamma)$  have  $\Delta(f) = \Delta_{\text{Zh}}(f)$ .

*Proof:* Let  $f \in \text{Zh}(\Gamma)$  and  $X_f$  a vertex set for  $\Gamma$  such that  $f \in \mathcal{C}^2(\Gamma \setminus X_f)$ . To see that  $f \in \mathcal{D}(\Gamma)$  we need to show that  $d_{\vec{v}}f(p)$  exists for all  $p \in X_f$  and  $\vec{v} \in T_p(\Gamma)$ . Hence let  $p \in X_f$  and  $\vec{v} \in T_p(\Gamma)$  and let  $t_0 > 0$  such that  $p + t\vec{v} \in \Gamma \setminus X_f$  for all  $t \in (0, t_0)$ . Furthermore abuse notation by writing  $f(t)$  for  $f(p + t\vec{v})$  and observe that  $f \in \mathcal{C}^2((0, t_0))$ . Obviously  $d_{\vec{v}}f(p)$  exists if and only if  $\lim_{t \rightarrow 0^+} f'(t)$  exists. So let  $\epsilon > 0$  and choose  $0 < \delta < t_0$  in a way that  $|\int_{(0, \delta)} |f''(t)| dt| < \epsilon$ , which is possible as  $f'' \in L^1(\Gamma, dx)$ . Then for all  $t_1, t_2 \in (0, \delta)$  we get

$$(4) \quad |f'(t_2) - f'(t_1)| = \left| \int_{t_1}^{t_2} f''(t) dt \right| \leq \int_{t_1}^{t_2} |f''(t)| dt < \epsilon,$$

hence  $\lim_{t \rightarrow 0^+} f'(t)$  exists and  $f \in \mathcal{D}(\Gamma)$ .

Now let  $\{E_i\}_{i \in \mathbb{N}}$  be family of pairwise disjoint sets in  $\mathcal{A}$ . By [BR10, Prop. 3.5(B)] we can assume that  $E_i \in \mathcal{A}$  is connected and closed  $\forall i \in \mathbb{N}$ , hence even further we may assume that  $\{E_i\}_{i \in \mathbb{N}}$  consists of disjoint sets which are either a closed interval or an isolated point. For  $p \in \Gamma \setminus X_f$  have  $m_f(\{p\}) = -\sum_{\vec{v} \in T_p(\Gamma)} d_{\vec{v}}f(p) = 0$  as seen before, and for a closed interval  $[t_0, t_1]$  on an edge of  $\Gamma \setminus X_f$  obtain with (2) and (4)

$$|m_f([t_0, t_1])| = \left| \sum_{\substack{p \in b([t_0, t_1]), \\ p \in [t_0, t_1]}} \sum_{\vec{v} \in \text{Out}(p, [t_0, t_1])} d_{\vec{v}}f(p) \right| = |f'(t_1) - f'(t_0)| \leq \int_{t_0}^{t_1} |f''(t)| dt.$$

Using this obtain

$$\sum_{i \in \mathbb{N}} |m_f(E_i)| \leq \sum_{p \in X_f} |m_f(\{p\})| + \int_{\Gamma} |f''(t)| dt < \infty,$$

hence  $f \in \text{BDV}(\Gamma)$  as desired.

It remains to show that  $\Delta(f) = \Delta_{\text{Zh}}(f)$ . For this it suffices to show equality on points  $p \in X_f$  and open intervals  $(c, d)$  contained in an edge of  $\Gamma \setminus X_f$ . For  $p \in X_f$  get

$$\Delta(f)(\{p\}) = - \sum_{\vec{v} \in T_p(\Gamma)} d_{\vec{v}}f(p) = \Delta_{\text{Zh}}(f)(\{p\}).$$

For  $(c, d)$  as above get by (2) that

$$\begin{aligned}\Delta(f)((c, d)) &= m_f((c, d)) = \sum_{\substack{p \in b((c, d)), \\ p \notin (c, d)}} \sum_{\vec{v} \in \text{In}(p, (c, d))} d_{\vec{v}} f(p) \\ &= f'(c) - f'(d) = - \int_c^d f''(x) dx = \Delta_{\text{Zh}}(f)((c, d)).\end{aligned}$$

□

**1.4. Proposition.** Let  $f \in \text{BDV}(\Gamma)$  and assume

$$(5) \quad \Delta(f) = g(x)dx + \sum_{p_i \in X} c_{p_i} \delta_{p_i}(x)$$

for a piecewise continuous function  $g \in L^1(\Gamma, dx)$  and a finite set  $X \subseteq \Gamma$ . Furthermore let  $X_g \subseteq \Gamma$  be a vertex set containing  $X$  and the finitely many points where  $g$  is not continuous. Put  $c_{p_i} := 0 \forall p_i \in X_g \setminus X$ . Then the following holds:

- i.)  $f''(x) = -g(x) \forall x \in \Gamma \setminus X_g$ ,
- ii.)  $f \in \text{Zh}(\Gamma)$ ,
- iii.)  $\Delta(f)(\{p_i\}) = c_{p_i} \forall p_i \in X_g$ .

*Proof.* Consider an edge in  $\Gamma \setminus X_g$ , identifying it with an interval  $(a, b)$  via our chosen parametrization. For each  $x \in (a, b)$  have  $-\sum_{\vec{v} \in T_x(\Gamma)} d_{\vec{v}} f(x) = \Delta(f)(\{x\}) = 0$ , where the last equality follows from (5) as  $x \notin X$ , hence  $f'(x)$  exists.

For small  $h > 0$  get

$$\begin{aligned}f'(x+h) - f'(x) &= -(-(-f'(x) + f'(x+h))) \\ &= -(-(\sum_{\substack{p \in b([x, x+h]), \\ p \in [x, x+h]}} \sum_{\vec{v} \in \text{Out}(p, [x, x+h])} d_{\vec{v}} f(p))) \\ &= -\Delta(f)([x, x+h]) = - \int_x^{x+h} g(t) dt.\end{aligned}$$

Analogously for  $h < 0$  obtain  $f'(x+h) - f'(x) = -(-f'(x+h) + f'(x)) = \Delta(f)([x+h, x]) = \int_{x+h}^x g(t) dt = - \int_x^{x+h} g(t) dt$ . Hence

$$f''(x) = \lim_{h \rightarrow 0} \frac{f'(x+h) - f'(x)}{h} = \lim_{h \rightarrow 0} \left( -\frac{1}{h} \cdot \int_x^{x+h} g(t) dt \right) = -g(x),$$

which shows i.), while ii.) and iii.) are direct consequences. □

**1.5. Corollary.** If  $f \in \text{BDV}(\Gamma)$  and  $\Delta(f) = \sum_{i=1}^k c_i \delta_{p_i}$  is a discrete measure, then  $f \in \text{CPA}(\Gamma)$ .

*Proof.* Since  $\Delta(f)$  discrete, obtain by 1.4 ii.) that  $f \in \text{Zh}(\Gamma)$  and hence  $\Delta(f) = \Delta_{\text{Zh}}(f)$ . Fixing appropriate vertex set  $X$  for  $\Gamma$  we see by 1.4 i.) that  $f''(x) =$

$-g(x) = 0$  on  $\Gamma \setminus X$ , so  $f(x)$  is affine on each segment of  $\Gamma \setminus X \implies f \in \text{CPA}(\Gamma)$ .  $\square$

## 2. FINITE SIGNED BOREL MEASURES ON $\Gamma$

Our aim now is to show that every finite signed Borel measure on  $\Gamma$  of total mass 0 already is the Laplacian of some function in  $\text{BDV}(\Gamma)$ . We first remind of some measure-theoretic statements.

### 2.1. Reminder.

- i.) (Weak convergence) Let  $X$  be metric space with Borel  $\sigma$ -algebra  $\Sigma$ . We say that a sequence  $\{\mu_n\}$  of Borel measures *converges weakly* to Borel measure  $\mu$  if for every  $f \in \mathcal{C}_{\text{bd}}(X)$  have

$$\lim_{n \rightarrow \infty} \int_X f d\mu_n = \int_X f d\mu.$$

Analogously define weak convergence for signed Borel measures.

- ii.) (Hahn decomposition) Let  $\mu$  be finite signed measure on measurable space  $(X, \Sigma)$ . There exist two measurable sets  $P, N$  such that
- $P \cup N = X$  and  $P \cap N = \emptyset$ ,
  - $\mu(E) \geq 0 \forall E \in \Sigma$  with  $E \subseteq P$ ,
  - $\mu(E) \leq 0 \forall E \in \Sigma$  with  $E \subseteq N$ .

We get (nonnegative) measures  $\mu^+$  and  $\mu^-$  by  $\mu^+(E) = \mu(P \cap E)$  and  $\mu^-(E) = \mu(N \cap E) \forall E \in \Sigma$ .

Both  $\mu^+$  and  $\mu^-$  are finite (nonnegative) measures and satisfy

$$(6) \quad \mu = \mu^+ - \mu^-.$$

The measure  $|\mu| = \mu^+ + \mu^-$  is the *variation* of  $\mu$  and  $|\mu|(X)$  is called the *total variation* of  $\mu$ .

- iii.) With (6) one can show a "triangle inequality"

$$(7) \quad \begin{aligned} \left| \int f d\mu \right| &= \left| \int f d\mu^+ - \int f d\mu^- \right| \\ &\leq \left| \int f d\mu^+ \right| + \left| \int f d\mu^- \right| \\ &\leq \int |f| d\mu^+ + \int |f| d\mu^- \\ &= \int |f| d|\mu|. \end{aligned}$$

**2.2. Definition.** Let  $\nu$  be finite signed Borel measure on  $\Gamma$ . A sequence  $\{\nu_n\}_{n \in \mathbb{N}}$  of finite signed Borel measures converges *moderately well* to  $\nu$  if:

- (A) There is bound  $B > 0$  such that  $|\nu_n|(\Gamma) \leq B \forall n \in \mathbb{N}$ .

(B) For each segment  $D \subseteq \Gamma$  (open, closed, half-open) we have  $\lim_{n \rightarrow \infty} \nu_n(D) = \nu(D)$ .

2.3. **Remark.** Let  $\nu$  and  $\{\nu_n\}$  as in Def. 2.2.

i) As each set in  $\mathcal{A}$  is finite disjoint union of segments, condition (B) implies that

$$(8) \quad \lim_{n \rightarrow \infty} \nu_n(S) = \nu(S) \quad \forall S \in \mathcal{A},$$

in particular  $\lim_{n \rightarrow \infty} \nu_n(\Gamma) = \nu(\Gamma)$ , and  $|\nu|(\Gamma) \leq B$ .

ii) By construction of appropriate step functions using characteristic functions of elements in  $\mathcal{A}$ , we obtain for  $f \in \mathcal{C}_{\text{bd}}(\Gamma)$  with (8) that  $\{\nu_n\}$  converges weakly to  $\nu$ .

iii) For any finite signed Borel measure  $\nu$  on  $\Gamma$  there is a sequence of discrete signed measures which converges moderately well to  $\nu$ . For details of the construction see [BR10, Section 3.6, p.63].

We can finally state our main proposition.

2.4. **Proposition.** Let  $\nu$  be finite signed Borel measure on  $\Gamma$ . Fix  $z \in \Gamma$  and put  $h(x) = \int_{\Gamma} j_z(x, y) d\nu(y)$ . Let  $M = |\nu|(\Gamma)$  be the total variation of  $\nu$ . Then:

- i.) Have  $h \in \text{BDV}(\Gamma)$  and  $\Delta(h) = \nu - \nu(\Gamma)\delta_z$ .
- ii.) For each  $x \in \Gamma$  and each  $\vec{v} \in T_x(\Gamma)$  have  $|d_{\vec{v}}h(x)| \leq M$ .
- iii.) Let  $\{\nu_n\}_{n \in \mathbb{N}}$  be any sequence of finite signed Borel measures which converges weakly to  $\nu$ . For each  $n \in \mathbb{N}$  put  $h_n(x) = \int_{\Gamma} j_z(x, y) d\nu_n(y)$ . Then  $\{h_n\}_{n \in \mathbb{N}}$  converges pointwise to  $h$  on  $\Gamma$  and if there is  $B \geq 0$  such that  $|\nu_n|(\Gamma) \leq B$  for all  $n \in \mathbb{N}$ , the convergence is uniform.
- iv.) If  $\{\nu_n\}_{n \in \mathbb{N}}$  converges moderately well to  $\nu$ , then for each  $x \in \Gamma$  and  $\vec{v} \in T_x(\Gamma)$ ,

$$(9) \quad \lim_{n \rightarrow \infty} d_{\vec{v}}h_n(x) = d_{\vec{v}}h(x).$$

2.5. **Remark.** Statement iv.) need not hold if  $\{\nu_n\}_{n \in \mathbb{N}_{\geq 1}}$  merely converges weakly to  $\nu$ . For example let  $\Gamma = [0, 1]$ ,  $z = 0$ ,  $\nu = \delta_1 - \delta_0$  and let  $\nu_n = \delta_1 - \delta_{\frac{1}{n}}$  for each  $n \geq 1$ . Then  $\{\nu_n\}$  converges weakly to  $\nu$ , but not moderately well, as for  $(0, 1] \subseteq \Gamma$  we have  $\nu((0, 1]) = 1 \neq 0 = \lim_{n \rightarrow \infty} \underbrace{\nu_n((0, 1])}_{=0}$ .

Use the explicit construction of  $j_z(x, y)$  as in [BR10, Section 3.3, p.52] and obtain

$$h(x) = \underbrace{j_0(x, 1)}_{=x} - \underbrace{j_0(x, 0)}_{=0} = x$$

and analogously

$$h_n(x) = \underbrace{j_0(x, 1)}_{=x} - \underbrace{j_0(x, \frac{1}{n})}_{\substack{=x \text{ if } x < 1/n, \\ 1/n \text{ else}}} = \max(0, x - \frac{1}{n}).$$

However for the unique  $\vec{v} \in T_0(\Gamma)$  we have  $d_{\vec{v}}h(0) = 1$ , while  $d_{\vec{v}}h_n(0) = 0 \forall n \geq 1$ .

**2.6. Corollary.** If  $\nu$  is finite signed Borel measure on  $\Gamma$  with  $\nu(\Gamma) = 0$ , then there exists  $h \in \text{BDV}(\Gamma)$  such that  $\Delta(h) = \nu$ .

*Proof.* Follows immediately from 2.4, part i).  $\square$

**2.7. Corollary.** Let  $\nu$  be finite signed Borel measure on  $\Gamma$ , let  $y \in \Gamma$  and consider

$$F_y(x) := j_\nu(x, y) := \int_{\Gamma} j_\xi(x, y) d\nu(\xi).$$

Then  $F_y \in \text{BDV}(\Gamma)$  satisfying  $\Delta_x(F_y) = \nu(\Gamma)\delta_y - \nu$ .

*Proof.* From [BR10, Prop. 3.3(A)] we see that for any  $z \in \Gamma$ ,

$$\begin{aligned} F_y(x) &= \int_{\Gamma} j_\xi(x, y) d\nu(\xi) \\ &= \int_{\Gamma} j_z(x, y) - j_z(x, \xi) - \underbrace{j_z(y, \xi) + j_z(\xi, \xi)}_{=: C < \infty, \text{ independent of } x} d\nu(\xi) \\ &= \nu(\Gamma)j_z(x, y) - \int_{\Gamma} j_z(x, \xi) d\nu(\xi) - C. \end{aligned}$$

With 2.4 and as  $\Delta_x j_z(x, y) = \delta_y - \delta_z$  obtain

$$\Delta_x(F_y) = \nu(\Gamma)(\delta_y - \delta_z) - (\nu - \nu(\Gamma)\delta_z) = \nu(\Gamma)\delta_y - \nu.$$

$\square$

**2.8. Proof of Proposition 2.4.** Fix  $z \in \Gamma$  and put  $h(x) = \int_{\Gamma} j_z(x, y) d\nu(y)$ . We first show that  $h \in \mathcal{D}(\Gamma)$ , i.e. need to show that  $d_{\vec{v}}h(x)$  exists for each  $x \in \Gamma$  and  $\vec{v} \in T_x(\Gamma)$ . Observe that for such  $x, \vec{v}$  have

$$(10) \quad d_{\vec{v}}h(x) = \lim_{\tau \rightarrow 0^+} \int_{\Gamma} \frac{j_z(x + \tau\vec{v}, y) - j_z(x, y)}{\tau} d\nu(y),$$

provided the limit exists.

Let  $S$  be a vertex set for  $\Gamma$  and consider  $\tau$  small enough that  $x + \tau\vec{v}$  lies on the edge of  $\Gamma \setminus (S \cup \{x, z\})$  in direction of  $\vec{v}$ . Let w.l.o.g.  $e_\tau = (x, x + \tau\vec{v})$  be the open segment contained in that edge. By [BR10, Prop. 3.3(A)] the function  $t \rightarrow j_z(t, y)$  is continuous in  $t$  and affine on edges of  $\Gamma \setminus (S \cup \{y, z\})$  (in particular the slope is constant there). So for  $y \notin e_\tau$  we have  $(j_z(x + \tau\vec{v}, y) - j_z(x, y))/\tau = \partial_{x, \vec{v}} j_z(x, y)$ . This implies that

$$(11) \quad \begin{aligned} & \int_{\Gamma} \frac{j_z(x + \tau\vec{v}, y) - j_z(x, y)}{\tau} d\nu(y) \\ &= \int_{\Gamma \setminus e_\tau} \partial_{x, \vec{v}} j_z(x, y) d\nu(y) + \int_{e_\tau} \frac{j_z(x + \tau\vec{v}, y) - j_z(x, y)}{\tau} d\nu(y). \end{aligned}$$

If  $y \in e_\tau$ , [BR10, Prop. 3.3(A)] gives  $|(j_z(x + \tau\vec{v}, y) - j_z(x, y))/\tau| \leq 1$  as  $\rho(x + \tau\vec{v}, x) = \tau$ . If  $y \notin e_\tau$ , [BR10, Prop. 3.3(D)] gives  $|\partial_{x,\vec{v}}j_z(x, y)| \leq 1$ . Hence as  $\tau \rightarrow 0^+$  the first integral in (11) converges to  $\int_\Gamma \partial_{x,\vec{v}}j_z(x, y)d\nu(y)$ , while the second one is bounded by  $|\nu|(e_\tau)$  and hence converges to 0. Thus the limit in (10) exists and we obtain

$$(12) \quad d_{\vec{v}}h(x) = \int_\Gamma \partial_{x,\vec{v}}j_z(x, y)d\nu(y).$$

Using again that  $|\partial_{x,\vec{v}}j_z(x, y)| \leq 1 \forall y \in \Gamma$ , we at once get  $|d_{\vec{v}}h(x)| \leq |\nu|(\Gamma) = M$ , which proves ii.).

Now let  $\{\nu_n\}$  be any sequence of finite signed Borel measures converging weakly to  $\nu$  and put  $h_n(x) := \int_\Gamma j_z(x, y)d\nu_n(y)$ . For each  $x$  the kernel  $F_x(y) = j_z(x, y)$  is continuous in  $y$ , nonnegative and bounded by [BR10, Prop. 3.3(A)], so  $\{h_n\}$  converges pointwise to  $h$  just by definition of weak convergence in Reminder 2.1 i.). Also by [BR10, Prop. 3.3(A)] we have  $|j_z(x_1, y) - j_z(x_2, y)| \leq \rho(x_1, x_2) \forall x_1, x_2 \in \Gamma$ , so if there is bound  $B$  such that  $|\nu_n|(\Gamma) \leq B$  for all  $n$ , we obtain

$$|h_n(x_1) - h_n(x_2)| \stackrel{2.1 \text{ iii.)}}{\leq} \int_\Gamma |j_z(x_1, y) - j_z(x_2, y)| d|\nu|(y) \leq B \cdot \rho(x_1, x_2),$$

and the functions  $h_n$  are all bounded by the same Lipschitz constant. As  $\Gamma$  is compact, by standard calculus the convergence of  $\{h_n\}$  to  $h$  is uniform, which shows iii.).

For part iv.) assume that  $\{\nu_n\}$  converges moderately well to  $\nu$ . Let  $x \in \Gamma$  and  $\vec{v} \in T_x(\Gamma)$ ; we need to show

$$\lim_{n \rightarrow \infty} d_{\vec{v}}h_n(x) = d_{\vec{v}}h(x),$$

or equivalently using (12),

$$(13) \quad \lim_{n \rightarrow \infty} \int_\Gamma \partial_{x,\vec{v}}j_z(x, y)d\nu_n(y) = \int_\Gamma \partial_{x,\vec{v}}j_z(x, y)d\nu(y).$$

We don't give a full proof, just a short note: The difficulty is that  $\partial_{x,\vec{v}}j_z(x, y)$  need not be continuous and  $\nu$  and  $\nu_n$  might have point masses. However the conditions of moderately well convergence allow us to construct appropriate step functions to show (13). For full details see [BR10, Section 3.6, pp.65-66].

It remains to show part i.). By Remark 2.3 iii.) we can choose sequence of discrete signed measures  $\{\nu_n\}$  converging moderately well to  $\nu$ . Use notation of  $h_n$  as above. By definition of  $m_{h_n}$  and  $m_h$  as in (2) and by (9) we see that each  $S \in \mathcal{A}$  satisfies

$$\lim_{n \rightarrow \infty} m_{h_n}(S) = m_h(S).$$

For  $n \in \mathbb{N}$  denote  $\nu_n = \sum_{i \in \mathbb{N}} \lambda_{i,n} \delta_{c_{i,n}}$  for  $\lambda_{i,n} \in \mathbb{R}, c_{i,n} \in \Gamma$ . Then

$$h_n(x) = \int_\Gamma j_z(x, y)d\nu_n(y) = \sum_{i \in \mathbb{N}} \lambda_{i,n} j_z(x, c_{i,n})$$

and hence

$$\begin{aligned}\Delta(h_n) &= \sum_{i \in \mathbb{N}} \lambda_{i,n} (\delta_{c_{i,n}} - \delta_z) \\ &= \sum_{i \in \mathbb{N}} \lambda_{i,n} \delta_{c_{i,n}} - \delta_z \sum_{i \in \mathbb{N}} \lambda_{i,n} \\ &= \nu_n - \nu_n(\Gamma) \cdot \delta_z.\end{aligned}$$

So  $m_{h_n}(S) = \Delta(h_n)(S) = \nu_n(S) - \nu_n(\Gamma)\delta_z(S) \forall S \in \mathcal{A}$ . Passing to  $n \rightarrow \infty$  yields  $m_h(S) = \nu(S) - \nu(\Gamma)\delta_z(S)$ . For countable family  $\{S_i\}$  of disjoint sets in  $\mathcal{A}$  it follows then that

$$\sum_{i \in \mathbb{N}} |m_h(S_i)| \leq 2|\nu|(\Gamma),$$

so indeed  $h \in \text{BDV}(\Gamma)$ .

The signed measure  $\Delta(h) = m_h^*$  attached to  $h$  is determined by its values on sets in  $\mathcal{A}$ , hence it must coincide with  $\nu - \nu(\Gamma)\delta_z$ . This finishes the proof.  $\square$

#### REFERENCES

- [BR10] M. Baker and R. Rumely. *Potential Theory and Dynamics on the Berkovich Projective Line*, Mathematical Surveys and Monographs, 159. American Mathematical Society, Providence, RI, 2010.